AN INTRODUCTION
TO THE PRINCIPLES
OF MECHANICS

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CONTENTS

PREFACE

MECHANICS is not a subject which appeals readily to the average student. He is apt to consider it a branch of science of unusual difficulty and incurable dullness. Doubtless there is something in the intrinsic quality of the subject which partly accounts for this, but one cannot help suspecting that much of the distaste evinced by students has its origin in the employment of unsuitable teaching methods. As Applied Mathematics or Theoretical Mechanics, it is presented in a way which is too academic and formalistic for the majority of students : moreover many are particularly handicapped for this method by inadequate mathematical training. On the other hand, when the subject is taught as Applied Mechanics there is almost inevitably a tendency unduly to emphasize practical details at the expense of principles.

This treatment of the subject, as either a branch of Mathematics, on the one hand, or of Workshop Practice, on the other, is an artificial and unsatisfactory arrangement, and must be held largely to blame for the defective results achieved. Mechanics should be regarded as a distinct branch of science, and taught as such. Students should realize that, although it is closely related to Mathematics and Physics, it has its own special fields of activity and its own characteristic laws and principles. The usual sharp division between Statics and Dynamics is to be regretted, as it tends to obscure the essential unity of the subject.

Mechanics, although based upon experiment, is very largely built up by a process of logical reasoning, and it is therefore a subject in which it is especially important that students should *think* and not merely *learn*. Far

too many rest content with acquiring little more than a medley of facts and formulæ which they are unable to use intelligently.

The experimental work, on which so much reliance is now placed, seldom achieves the results intended. There is a decided tendency to regard work in the mechanical laboratory as a piece of educational ritual, obligatory but not very interesting. Too often students treat it as routine work, to be struggled through by following imperfectly comprehended instructions. Moreover, there is something unconvincing about many of the usual experiments, and it is hardly surprising if they are not taken very seriously. Fewer and simpler experiments, done more slowly and thoroughly, might give better results; but it needs unusually favourable conditions, and particular care and skill on the part of tutors and demonstrators, if the avowed object of "making students think for themselves" is to be attained.

It has been my endeavour in this volume to present the fundamental principles of Mechanics in such a way that they may be realized as distinctive truths, having their definite and logical place in science. I have attempted to avoid, on the one hand, confusing and disheartening the student by excessive "book-work"—definitions, theorems, proofs, corollaries and formulæ—and, on the other hand, obscuring the reasoning by superfluous details of machines and structures, and other extraneous matter.

For the same reason, certain constructions, proofs and other matters usually treated in the body of the text, are here relegated to the appendix, and a number of minor applications are given only in the form of worked examples. Also the numerical examples have been collected together at the end of each chapter, instead of scattering them through the text. When a student wishes to work examples he will not be discouraged from so doing by the fact that he has to turn over a few pages to find them: nor, on the other hand, will he gain much from those thrust in his way when

he does not want them. The practice of interspersing the text with numerical examples, worked out in detail, prevents any continuity of thought, is intensely irritating to many students, and makes a book almost useless for purposes of reference or revision.

Absolute units of force are employed throughout the work for theoretical purposes, and in many examples. Experience has convinced me that much confusion of thought with regard to mass, weight and other matters, is due to the employment of gravitation units of force, and the so-called "engineer's unit" of mass. The use of absolute units is more logical and consistent and, once the idea is grasped, is simpler and easier. Any slight initial difficulty is very quickly overcome, and is more than compensated by the greater clearness and certainty obtained. In practical problems it is extremely easy to convert results into gravitation units when required.

I have tried especially to present the work as systematically as possible. Too often such fundamental ideas as the Parallelogram Law, moments in general, and the two kinds of motion, are given to the student piecemeal, so that he fails to realize the orderliness and symmetry of the subject: its continuity is lost to him because his attention is never explicitly drawn to it. It is unreasonable to expect the beginner to trace the connexion between isolated propositions for himself—the teacher who neglects to point it out can hardly be surprised if his students "cannot see the wood for the trees." For this reason I have tried to emphasize classification and comparison: first moments and second moments, vectors and scalars, translation and rotation, and so on. Also I have divided vector quantities into two classes: an innovation which is, I think, amply justified.

It is my hope that, by a careful study of this book, any student of ordinary capacity will obtain a clear conception of mechanical principles and a firm grasp of the most important methods of applying them. He will, of course, still have much to learn of the subject

—for this is only an Introduction—but if he has laid a sound foundation further progress will be easy.

I gratefully acknowledge the many valuable criticisms and suggestions I have received from Dr. William Garnett, M.A., D.C.L. The notification of any errors which may have escaped attention will be much appreciated.

J. F. S. ROSS.

LOUGHBOROUGH, 1922.

INTRODUCTION

AS the Author points out in his Preface, the teaching of Mechanics in "the Schools" has tended to fall between two stools. Either it has been treated in a purely deductive manner as a branch of mathematics and the whole subject developed from a few axioms, such as the "transmissibility of force," together with certain experimental results obtained by Galileo, Huyghens, Newton and others, and formulated as concise statements known as the Laws of Motion; or, the subject has been taught experimentally, and under the teacher's guidance pupils have been led to verify, or discover for themselves, experimental results which are consistent with the fundamental "laws" of mechanics and thus to show the probability of the truth of those "laws." As an example of the former method Thomson and Tait's *Natural Philosophy* affords a good illustration of the deductions which used to be drawn from such a simple statement as "Action and reaction are equal and opposite."

The great development of the experimental study of mechanics among young pupils resulted from the Technical Instruction Acts under which laboratories were provided by Local Authorities in most Secondary and many Higher Elementary schools. Unfortunately, the conclusions derived theoretically in the text-books on mechanics were, in many cases, not so closely allied to the results obtainable in the laboratory by elementary methods as was the case with some other branches of experimental science. When a beginner finds that with some "mechanical powers," and those the contrivances most used in practice, more than 50 per cent. of the work he does is transformed by friction, and sometimes very much more, he is inclined to lose faith altogether in mechanics as "a subject." The mechanical

laboratory has been a success in the schools only when the teacher has been specially well equipped so that "theory" and "practice" have been promptly reconciled. Partly in consequence of these difficulties, electricity, the laws of which are vastly simpler than those of mechanics, or some other branch of experimental physics, especially heat, has generally been preferred to mechanics as a subject of school study.

The mathematical treatment of mechanics was hindered in the schools by the habit of teaching mathematics in compartments, each taken in regular order—Euclid, Algebra, Trigonometry, Plane Co-ordinate Geometry, Differential and Integral Calculus and so on, all applications of mathematics to natural phenomena being postponed until the analytical weapons were complete. Many a pupil who would have been interested in mechanics, hydrostatics and the application of elementary mathematics to problems in physics did not appreciate the long preliminary training in pure mathematics and failed to make the progress of which he was capable. Hence it came about that among the students entering engineering colleges, when these were established, few had any knowledge of mechanics and many were without the necessary mathematical equipment. In this position of affairs Professor John Perry came to the rescue with his system of teaching mathematics to engineering students. In the hands of a man of Professor Perry's ability his method could command success. It provided the student with just the amount of mathematical theory which his problems required; but with teachers of less ability the imitation of Professor Perry's methods, while appearing to offer a royal road to the solution of mechanical problems, proved to be lacking in the power of affording the mental training requisite to enable the student to grapple successfully with problems differing in type from those worked out in his text-book or by his teacher.

Captain Ross's book is sound in its principles and methods. The student is led along a gradually-ascend-

ing path, but each step is clearly cut on a solid foundation. There are no breaches of continuity, no crevasses to be cleared by a flying leap, and no obstacles to be surmounted which require the use of the guide's rope. In nearly all cases the complete proof of the accuracy of the method is set out in the text and some very ingenious methods are given to enable results to be obtained without reference to the integral calculus. In a few, but very few, cases the analytical proof is relegated to the Appendix. The method which the Author has adopted appears to be thoroughly scientific and well adapted to the needs of the great majority of engineering students. On the one hand, he brings the student at once into the heart of his subject without running the risk of tiring him with an array of theorems of purely academic interest. On the other hand, he never supplies formulæ or rules without firmly establishing them on fundamental principles. One of the most satisfactory features in the book is the systematic use of absolute units. These units are employed in every other branch of science, and they have the great advantage of being independent of the variation of gravity over the earth's surface or elsewhere. In this respect there is a broad distinction to be drawn between the relation of the absolute system to the gravitation system and the relation to one another of two absolute systems based on different fundamental units, such as the foot-pound-second and centimetre-gramme-second systems. In the latter case there is a fixed numerical relation between the derived units of the same dimensions. In the former no such fixed relation exists, the ratio depending on the acceleration of gravity at the place. If the student adopting the foot-pound-second system of absolute units in all his dynamical problems will remember that in his results all expressions for force, momentum, energy, etc., are always in absolute units and must be divided by the numerical value of g if the results are to be expressed in gravitation units, there need be no difficulty whatever in presenting the

results in the form in which they are required and no doubt in the student's mind with regard to the units to which his calculations refer.

The conception of mass as that property in virtue of which a finite force is necessary to produce a finite acceleration is a less familiar conception than that of weight, of which we are all conscious from infancy, and as we never have the opportunity of handling things unexposed to the earth's attraction, we do not realize that weight is an exoteric property of a body due to the action of external causes, while mass is an esoteric or intrinsic property inherent in the body itself. Nearly fifty years ago Cambridge students had presented to them in class an opportunity of studying mass almost irrespective of weight. A half-hundredweight was suspended by a rope about 30 feet in length and it was shown that a pull of two pounds' weight would deflect it by more than a foot from its position of rest. The students were invited to seize the suspended weight by both hands and to swing it backwards and forwards through a couple of feet as rapidly as they could without making any attempt to relieve the rope of its burden. The operator became quickly exhausted in this experiment, the cause of the exhaustion being the mass, not the weight, of the half-hundredweight. Mass being thus an intrinsic property, it is very undesirable that its units should depend on external conditions, e.g. g pounds.

The study of the dimensions of derived units leads to a test of the accuracy of the work in the solution of a dynamical problem which is almost as useful as the casting out of nines in arithmetic, for if all the terms on both sides of an equation, or all the terms in any compound expression, are not of the same dimensions in the fundamental units, it is certain that a mistake has been made.

There are some who speak with great authority and hold that for the purposes of the engineer gravitation units will always be employed, and the convenience of

making dynamical calculations involving mass, momentum, energy, etc., in terms of the pound weight as the fundamental unit from which the unit of mass of g pounds' weight is derived, more than counterbalances the inconvenience of employing a variable unit of mass, for the engineer is much more directly concerned with forces than with masses. But the gravitation unit was the cause of much of the confusion which overshadowed the subject of dynamics sixty or seventy years ago, and when a system of units of universal application in work of extreme accuracy was required for the development of electrical science, there was no question of the advantages afforded by an unchangeable unit of mass. Because the engineer conducts his calculations respecting moving bodies in absolute units he is not precluded from expressing steam pressure in pounds' weight per square inch, or stresses in elements of a structure in tons' weight per square inch. Errors likely to arise from other causes are greater than the variation of gravity if steel is tested in England and used for a bridge over the Congo.

The Author has been particularly happy in his treatment of the Gyroscope and allied problems. The theory of Precession is not generally dealt with in elementary works on Mechanics, but under the guidance of Captain Ross the student finds that he has acquired a knowledge of the subject without being conscious of any of its difficulties. Uniform rotation about an axis, while the axis itself is rotating in its own plane or is describing a cone, involves questions of relativity which have only recently become an important study for the engineer. In nearly all the work of the mechanical engineer axes of rotation may be regarded as fixed, for though they generally participate in the earth's diurnal rotation that motion is too slow to make itself felt in ordinary workshop practice. But submarine navigation has been made possible through the high-speed gyroscope keeping its axis of rotation fixed in direction, notwithstanding the revolution of its frame about the earth's polar axis, and

appreciable stresses are introduced into the shaft of a running marine turbine, when the ship is pitching in a sea-way, which would not be there if the engine were not rotating. The Bessemer Saloon was one of the earliest applications of the principle of the gyroscope to practical ends. To the meteorologist the earth's rotation is of primary importance in all questions of the movements of large masses of air and the production of winds and cyclones.

The main object of this Introduction is to urge the teacher to warn the student against the danger of employing "rules," whether of the nature of algebraical formulæ or methods of geometrical construction, for the solution of practical problems unless he is sufficiently acquainted with the principles on which the solution is based to enable him, by an independent calculation, to arrive at an approximate result sufficiently near to afford an assurance that he has not made a serious error in the application of his formula or geometrical construction. As the Author says in his Preface, "students should *think*, not merely *learn*." It is now generally recognized that a student, before he is satisfied with the answer he has worked out to a complicated arithmetical question, should check it by the application of "common sense" and a rough mental calculation which will at least show him if he has made a mistake in the position of a decimal point, or mistaken feet for inches in some algebraical formula, or obtained a result in absolute units instead of gravitation units. Certainly no designer should be entrusted with the preparation of designs for construction unless he is thoroughly acquainted with the principles on which his practical and time-saving methods are based; so that he can always check his work without recourse to his formula or his geometrical method. About thirty years ago, a very responsible firm of steel constructors accepted a contract for the steel framing of a roof of wide span. Provided that the roof principals could be accommodated in the space arranged for them between the slates and

the ceiling and would sustain a certain specified "safe load," the details of the design were left to the contractors. When the principals were in position on the walls, it was noticed that the upper and lower members could resist only half the stresses which they were required to sustain at the points in their length where these stresses were greatest, and certain members had been designed for tension which were actually in compression. As the contractors were *bonâ fide* satisfied that the work was far stronger than the specification required, it was resolved to test one of the principals *in situ* with a uniformly distributed load suspended from the crossings of the purlins. At little over half the specified safe load the elastic limit was reached, and at less than two-thirds the principal spread so as to crack one of the sustaining walls, while the particular members mentioned above took the form of an elongated letter S. The contractors then, as evidence of their *bonâ fides*, submitted to the building owners the stress diagram from which the scantlings had been calculated, and a minute's inspection showed that though all the loading was vertical the horizontal components of the stresses across a complete vertical section of the frame did not balance. This was a certain proof of the existence of some error in the drawing, and it did not then require much further inspection to find where it originated. A single slip on the part of the draughtsman had done all the mischief: but if he had been accustomed to test his results by such rough calculations as have been indicated, or by the simple test of the balancing of the horizontal stresses on a complete vertical section, the whole trouble, and the risk of constructing a roof which might at any time have collapsed on a public audience, would have been avoided. The first accident to the Quebec Bridge in August, 1907, when nearly 15,000 tons of steel collapsed, though the structure was carrying only three-quarters of its ultimate dead load, may possibly afford an example of a very similar error but on a much greater scale. There was

considerable resemblance between the methods of procedure in the two cases.

Another piece of advice which the teacher may well press upon his students when they are about to enter engineering works, is that they should pay due respect to the men who have devoted years to practical work but have not had the advantage of a college training. Some years ago, there were complaints that college students gave the impression to their fellow-workmen, who, no doubt, were hyper-sensitive, that they thought that they alone understood their subject. There can be no greater mistake than to look askance on the man who has spent a great part of his life in the shop. His experience and his trained eye are likely to be a surer guide than all the knowledge the student has acquired from classes and text-books, until that knowledge has been blended with experience. When that combination has been achieved the man with the theoretical training has vast advantages over the man of practice only, and can successfully attack new problems which are quite beyond the powers of the other. But there are very many points to be observed and details to be taken into account which cannot be described in text-books or lectures and can only be known by experience. An able student, an apprentice in a locomotive works, had a difference with the shop foreman about the proper diameter of a weigh-shaft—the shaft which transmits the pull of the brake lever to the brakes. The student's estimate was $4\frac{1}{2}$ inches. The foreman contended that $3\frac{1}{2}$ inches would suffice. The difference in the torsional strength was as the cubes of these diameters, practically 2 to 1. The student brought his calculation to his teacher, who, during the class meeting, could find no error in it, though he had difficulty in believing that the foreman would be in error to so large an extent: but reflecting on the subject as he walked home he realized that the pull on the brake lever had to be divided between two sets of brakes, one-half being transmitted to the right and the other to the left along the weigh-shaft, so that the actual torsion

was only one-half of that for which the calculation had been made. Here was the explanation, and the foreman won. An absurd blunder, the reader will say, and discreditable to the teacher; but most blunders appear absurd when they have been exposed.

On another occasion a draughtsman in a marine engine works brought to his class teacher a new design of boiler, which his firm was proposing to manufacture. The draughtsman had discussed the design with the "Boss" and had expressed the opinion that under pressure the shell would hunch and that certain cross stays were required, but he could not convince his principal, who feared that other manufacturers would laugh at him for introducing unnecessarily a supplementary system of stays. The draughtsman's practical eye had not deceived him, though he could not put forward his arguments in a manner which would carry conviction. *Experientia docet.* The additional stays were wanted and the teacher was able to set forth a case.

Experientia docet. What advantage, then, have college students? Much every way, but chiefly that they have learned to observe intelligently, to think accurately, and to express themselves clearly. Clear expression and accurate thinking are very closely connected; for it is generally only when we attempt to put our thoughts into forms capable of expression that we think definitely and quantitatively. Nearly every one is more or less accustomed to think quantitatively in questions of money, though many fail to appreciate the relation between pence and pounds or larger units. But in other matters quantitative thinking is generally the result of special study (which money naturally secures through self-interest). The skilled workman, trained only in the shop, may be marvellously clever with his tools and may be able to work to a degree of accuracy almost inconceivable by the layman, but he has never acquired the habit of applying number and measure and weight to the subjects of everyday workshop experi-

ence; and outside his own job he is unable to express his thoughts clearly, even to himself.

Forty years ago, long before the introduction of Vanadium steel and tools which would take a half-inch cut from chrome steel at 250 feet a minute and preserve their cutting edges at a red heat, a new engineering workshop was established in a college, and the teacher, desirous of learning the local practice with regard to speed of cutting, called on the managing director of one of the foremost works in the district to inquire. The director was unable to answer his questions, but introduced him to the shop foreman, who described a nine-inch shaping machine taking a full cut on cast iron as going "Sh—sh—sh—sh—something like that," but could not express the speed numerically. He was of opinion that gun-metal was cut at the same speed, but was ready to give facilities for making measurements. The tools were found to be running at the orthodox speed for carbon steel, and when the managing director was asked how the diameter of the driving drum was determined for a new tool, his reply was that a pulley was fixed on the shaft on trial, and if the speed was wrong the pulley was changed. This was not an economical proceeding; it illustrated the lack of numerical data current among practical men at that time. But this was forty years ago, and under the influence of engineering colleges, standardization and scientific control, *nous avons changé tout cela*.

WM. GARNETT.

AN INTRODUCTION TO THE PRINCIPLES OF MECHANICS

CHAPTER I: PRELIMINARY

MECHANICS may be defined as the branch of science which deals with **Forces** and their effects. It is customary to divide it into several parts, and to treat each part as almost a separate subject: this has some advantages, but tends to conceal the intrinsic unity of the whole. We shall, therefore, ignore the divisions.

We have termed Mechanics a *branch* of science, rather than *a* science. It is important to realize that subjects such as Mechanics, Chemistry and Physics, whilst each having their own laws, methods and fields of activity, are nevertheless only parts of the much greater subject of Natural Science, and, as such, are closely related and to some extent overlap. The other branches of science to which Mechanics is most nearly related are Mathematics and Physics. Some knowledge of Mathematics is essential to a clear comprehension of the principles of Mechanics, and a knowledge of the elements of Physics, though less important, is also of service.

In the study of Mechanics the greatest difficulty which confronts us is that of obtaining a clear realization of the fundamental principles. These principles are few in number, and easily enumerated, but to know them by heart is not necessarily to understand them. Much of the uncertainty and hesitation felt by students in applying the laws and methods they have learnt, is due to the fact that they do not really comprehend them. We must make it our first endeavour, then,

thoroughly to master each new piece of work before passing on.

We have stated that the principles of Mechanics are few in number: wherein, then, lies the difficulty which many students find in trying to grasp them? One cause is certainly that, amongst the quantities with which we have to deal, are some which do not occur at all in everyday life, and cannot be explained in such a way that their nature is readily perceptible. Some of these quantities can only be expressed conveniently in mathematical form, and consequently the student, if unable to grasp their significance, is apt to take refuge in a mere memorization of their symbols and formulæ. Naturally, empirical knowledge of this sort is of little real value, and, when put to the test, breaks down and leaves the victim convinced that Mechanics is both incomprehensible and uninteresting.

We must, then, deliberately set before ourselves the complete mastery of *principles* as our definite aim, and must beware of entanglement with less important details. When we have really achieved this, we shall find no difficulty in dealing with minor points. We must be keenly on the alert to see that we do not delude ourselves into believing that we have grasped essentials, when we have, in fact, merely learnt by rote a few definitions and a formula or two. We must scrutinize closely each new statement and ponder over it until we have a firm grip of its meaning: we should then seek to apply it in a variety of ways, and note its bearing on other parts of the subject.

In doing this we must aim to express ourselves in words rather than symbols, and in the simplest and clearest language we can find. Carelessness in the choice of words is always deplorable: in science it is unpardonable. It should be our guiding rule never to employ symbols if we can satisfactorily dispense with them. They should be regarded as abbreviations, legitimately to be employed when they simplify argument or facilitate calculation, but having no virtue in themselves. Other-

wise they become a snare for the unwary, and a refuge for the incompetent.

Formulæ are in somewhat similar case: they are a necessary and useful means for expressing and recalling the relationship of mechanical quantities to each other, but they need to be kept in their proper place. Like so many other things, they are good servants but bad masters. The rule should be: never use a formula without both a clear understanding of its derivation, and the certainty of being able to prove it if required.

Little need be said here as to the use of graphical methods. Their value is now so well established that they are not likely to be ignored: students should, however, realize that the choice of a method must depend on the nature of the problem to be tackled, and that the best method is the one which is the most suitable for the particular circumstances. Hard and fast rules cannot be made: each case must be considered on its merits. The same caution is needed as with regard to symbols and formulæ. Graphical methods must never be used as a means of avoiding the necessity for thought. It is just as important to understand the reason for a graphical construction, and to be able to prove it, as to be clear about the derivation of a formula. Also, results obtained graphically should be checked by other methods.

METHOD OF STUDY. For students working independently, the following plan is suggested:—

(*a*) Read through a complete chapter carefully, but without spending too much time over it. This will give you an idea of the ground to be covered and the general lines of the argument.

(*b*) Re-read the chapter slowly, and endeavour to master the principles involved. Do not be content with vague ideas: if, after careful study, any point still presents difficulty to you, make a pencil note in the margin opposite to it, and take the first opportunity of getting it cleared up.

(*c*) Close the book and try to recall the reasoning

of the chapter, and the conclusions reached. Ask yourself questions about them, and try to find the answers.

(*d*) Without referring to the book, write out, as briefly as possible, and in your own words, a summary of the argument. Verify this afterwards by comparison with the book, and correct any errors.

(*e*) Turn to the worked examples at the end of the chapter, and go through them carefully, keeping in mind the principles involved, and noticing how they are applied in practice.

(*f*) Work through the remainder of the examples belonging to the chapter, and compare your results with the answers given at the end of the book. See hints on p. 21.

(*g*) Once more read through the chapter, and see that, as far as possible, no point remains uncomprehended.

(*h*) Repeat this process for each chapter in turn. Then finally go back to the beginning and read through the whole book again.

Students attending lectures on the subject may require to modify this plan to some extent, but should adhere to it as far as circumstances permit.

The principles of Mechanics are illustrated daily in a thousand and one incidents of ordinary life: the student should keep this in mind, and be constantly on the alert to apply them to everyday happenings. A grip of fundamental ideas, and a habit of noting their practical application—not under artificial laboratory conditions, but in common affairs—will make of Mechanics a living, fascinating subject.

Systematic work on these lines will give the student a real command of his subject in a surprisingly short time. If he has to prepare for a particular examination in Mechanics, he can then acquire any additional knowledge of formal theorems or practical details that the syllabus of the examination may necessitate, in the minimum of time, and without the danger of confusing then with essentials. Moreover, when he goes on to more advanced work, he will find that he has nothing

to unlearn, and that his grasp of fundamentals will carry him along with ease and pleasure, through work which others may find extremely perplexing.

UNITS. We *measure* any physical quantity, that is to say, we determine its size or *magnitude* by comparing it with some other quantity, of the same kind and of definite magnitude, which we have chosen as a standard. For example, we may measure lengths by comparing them with the particular length which we term a **foot.** We then say that a given length is equal to so many *feet.* Similarly, we may measure periods of time by comparing them with the particular period of time which we term a **second.** We then say that a given period of time is equal to so many *seconds.* Any such particular quantity, selected as a standard for measuring other quantities of the same kind, we term a **unit** of measurement.

Units are divided into two classes: **absolute** and **variable.** *Absolute units* are those which are exactly and definitely fixed, so that they are constant everywhere and at all times. For all scientific purposes absolute units are, of course, essential. *Variable units* are frequently used in everyday affairs, because of their practical convenience, and in spite of the fact that they are inexact and approximate. For example, we say that a certain distance is **three minutes' walk,** instead of saying that it is so many yards, although we are well aware that everybody does not walk at the same speed. Similarly, a doctor will give instructions to take a **teaspoonful** of medicine, although teaspoons are not all the same size. Clearly, such rough-and-ready units are quite unsuitable for scientific work.

Every unit must be either absolute or variable. But we can also classify units in various other ways. For example, certain absolute units have been selected by Parliament as the **legal standard units.** The legal standard unit of length in this country is the **yard,** which is the length of a certain bronze bar at a certain

temperature. The legal standard unit of mass is the **pound,** which is the mass of a certain lump of platinum. In Mechanics we have little direct concern with legal standard units.

For our purpose, a more important classification is that which divides units into the two classes of **fundamental** and **derived.** Let us see what we mean by these terms.

The three quantities **Mass, Length,** and **Time,** as we shall find later, are of primary importance. Now for each of these quantities a great variety of units is in use: so, in order to simplify and systematize our work, we choose *one* of the various units of mass, *one* of the units of length, and *one* of the units of time, and term them our three **fundamental units.** In each case the unit so chosen is an absolute one.

In the British system of measures we take the **pound** as the fundamental unit of mass, the **foot** as the fundamental unit of length, and the **second** as the fundamental unit of time.

In the French, or Continental, system of measures, which is used by scientists all over the world, the **gramme** has been chosen as the fundamental unit of mass, the **centimetre** as the fundamental unit of length, and the **second** as the fundamental unit of time.

Derived units are those obtained from one or more of the fundamental units. This derivation can be made in a great many different ways. For example, the area of a piece of ground 66 feet wide and 660 feet long is termed an **acre.** The acre is a derived unit, for it depends upon a fundamental unit, the *foot.* But we could obtain a unit of area from this same fundamental unit in a much simpler way. If we take an area whose length is **one** foot and whose width is also **one** foot, we have the simplest possible unit of area, viz., the **square foot,** that can be derived from the given fundamental unit.

We may term units which are derived from the fundamental units in the simplest possible way, **systematic units.** Obviously we can have *only one sys-*

tematic unit for each kind of quantity, when we are working with a given set of fundamental units.

The **foot,** the **pound,** and the **second,** together with all the *systematic* units derived from them, are known as the foot-pound-second, or **F.P.S.** system of units. Examples of F.P.S. systematic units are the **square-foot** which is the unit of *area,* and the **cubic-foot** which is the unit of *volume.*

The **centimetre,** the **gramme,** and the **second,** together with all the *systematic* units derived from them, are known as the centimetre-gramme-second, or **C.G.S.** system of units. Examples of C.G.S. systematic units are the **square-centimetre** and the **cubic-centimetre.**

Let us now sum up the classifications we have made.

ABSOLUTE units are those which always have the same value.

VARIABLE units are inexact and approximate, and are used, as a rule, for practical convenience where precise values are not required.

LEGAL STANDARD units are absolute units specified by law as standards of comparison.

FUNDAMENTAL units are arbitrarily chosen absolute units of *Mass, Length,* and *Time.*

DERIVED units are those obtained, in *any* way, from the fundamental units.

SYSTEMATIC units are those obtained in the *simplest* way from the fundamental units.

In this book we shall as far as possible employ absolute units, and particularly systematic units. The latter are the only units which have a consistent, logical relation to each other and to the fundamental units, so that they are in every way the best for theoretical work. But for practical purposes it is often convenient to use other units, e.g. in measuring long distances we may prefer to employ the *mile* as the unit of length, rather than the foot, and in measuring long periods of

time we prefer to employ bigger units than seconds; say hours or years.

In working exercises in Mechanics it is usually best to employ systematic units only, throughout all calculations. Even in problems where the data are given in practical units, and the results are required in practical units also, it is generally worth while to convert to systematic units before commencing the calculations, and at the conclusion to convert the results into whatever units may be most convenient. This plan may, at first sight, seem to involve unnecessary work, but it is in the long run an economical one, for it practically eliminates one of the most prolific sources of error and consequent waste of time and energy, namely confusion of units. Though its adoption may entail a little extra trouble initially, once the habit is formed it will be found greatly to facilitate clear thinking and accurate working, and so will more than repay the outlay.

DIMENSIONS. It can be shown that all the quantities with which we have to deal in Mechanics can be expressed in terms of **Mass, Length,** and **Time.** We therefore term these our three **fundamental quantities.** Let us consider a few examples.

A *line* has *length* only and no breadth or thickness. We say therefore that it is of **one dimension of length.**

An *area* has *length* and *breadth,* but no thickness. For example, the floor of a room is 16 feet in length and 12 feet in breadth, so that its area is 16 feet $\times$ 12 feet $=$ 192 square-feet. In order to determine this area we must know the two lengths, 16 feet and 12 feet. We say therefore that an area is of **two dimensions of length.**

A *volume* has *length, breadth,* and *thickness.* For example, if the room just considered is 10 feet in height, then the volume of the space enclosed is 16 $\times$ 12 $\times$ 10 $=$ 1,920 cubic-feet. In order to determine this volume we need to know the three lengths, 16 feet, 12 feet, and 10 feet. We say therefore that a volume is of **three dimensions of length.**

In studying Mechanics we shall have to consider a number of different quantities, some of which, such as lengths, areas, and volumes, are familiar quantities which we employ in ordinary everyday affairs, but others will be new to us, unless we have already studied the subject. Some of these new quantities are a little difficult to understand at first, and it is a help to so doing, if we are able to see how they compare with other and more familiar quantities. In order to facilitate the comparison we can consider the **dimensions** of the quantity with which we may be dealing, that is, we can see *how to express it in terms of the three fundamental quantities.*

To enable us to do this more readily, we usually employ a special system of symbols, one for each of the three fundamental dimensions. For one dimension of *mass* we use a capital M in square brackets, thus: **[M]**. For one dimension of *length* we use a capital L in square brackets, thus: **[L]**. For one dimension of *time* we use a capital T in square brackets, thus: **[T]**.

The dimensions of a straight line would therefore be denoted simply by: **[L]**.

The dimensions of an area would be denoted by $[L] \times [L]$ which we usually write in the form: $[L]^2$.

The dimensions of a volume would similarly be denoted by $[L] \times [L] \times [L]$ which we usually write in the form: $[L]^3$.

Density is defined as *mass per unit volume.* We obtain the density of a body by dividing its mass by the volume of space it occupies. The dimensions of density are therefore **[M]** divided by $[L]^3$, that is $\frac{[M]}{[L]^3}$, which we usually write in the form: $[M][L]^{-3}$.

The **specific gravity** of a substance is defined as the ratio of its density to the density of water. The dimensions of specific gravity are therefore $[M][L]^{-3}$, divided by $[M][L]^{-3}$, that is, $[M]^0[L]^0[T]^0$. *All* ratios, since they are merely numerical quantities, have zero dimensions, that is, they are *not* derived from the fundamental quantities of mass, length and time.

A period of time is of **one dimension of time** and is therefore denoted by: **[T]**.

We shall deal with the dimensions of other quantities, such as weight, speed, force, etc., later, and shall see how to express these quantities in terms of the three fundamental quantities. We shall also see that this system of dimensions has other advantages. It gives us a valuable means of checking the accuracy of our work in many cases, enabling us to detect errors that might otherwise be overlooked, and helping us to avoid them; which is even more useful. It can be utilised, moreover, to determine the relationship between certain quantities independently of our usual methods, and is in this way a serviceable adjunct to them.

The student should make sure that he has grasped the idea and understands clearly what is meant by the *dimensions* of a quantity. It is worth while to take a little trouble with this question, as it will be a great help to us later, especially in aiding us to keep our ideas clear and precise. Throughout his work in Mechanics the student should keep before him as his constant and deliberate aim, the attainment of **clear ideas**, and should never rest content with a vague understanding of any piece of work. It is not the *little* knowledge that is really the dangerous thing, so much as the indefinite and inaccurate knowledge which results from half-hearted, slovenly work.

SUMMARY OF CHAPTER I

Mechanics deals with **Forces** and their effects.

Complete mastery of *principles* should be the aim of all students of Mechanics. These principles are few in number but sometimes present difficulty: special attention must therefore be given to them.

The use of **symbols, formulæ,** and **graphical methods** enables us to shorten and simplify our work, but these aids must be employed intelligently and with discrimination.

It is advisable that the study of Mechanics should be conducted *systematically*. Practical applications should be noted, and many numerical examples worked.

All physical quantities can be expressed in terms of **Mass, Length,**

and **Time.** These are therefore termed the *three fundamental quantities.*

Units of measurement are either **absolute** or **variable** : the latter are unsuitable for scientific work.

Fundamental units are arbitrarily chosen absolute units of *Mass, Length,* and *Time.* **Derived** units are those obtained, in *any* way, from the fundamental units. **Systematic** units are those obtained in the *simplest* way from the fundamental units.

Two systems of measurement are in use in Mechanics, viz., the **F.P.S.** (foot-pound-second) and the **C.G.S.** (centimetre-gramme-second) systems.

The **dimensions** of a quantity are the three factors composing it, where each factor is one of the three fundamental quantities, raised to some integral power, which may be either positive or negative or may be zero.

One dimension of Mass is denoted by **[M]**.

One dimension of Length is denoted by **[L]**.

One dimension of Time is denoted by **[T]**.

The consideration of dimensions of quantities is an aid to clear thinking, and may be used to check the accuracy of much of our work.

HINTS ON WORKING EXAMPLES

1. Read through the whole question and see that you understand it before you commence your answer.

2. Whenever possible **make a sketch or diagram** and mark on it all the information given in the question. This will frequently make the question clearer, and will help you to decide on the best method to employ in answering it. Take care that the information is set down correctly and completely : it is always as well to check it through afterwards.

3. Insert sufficient explanations in your answers, to make it quite clear what you are doing and why. If you *must* use symbols, explain them. If you use a formula you should, if possible, prove it ; if you are unable to do this, at least explain it as far as you can.

4. Pay special attention to setting-out all the steps of your work as logically and clearly as possible. Bad writing and untidiness are prolific sources of error. All calculations and rough work should be done neatly, and if not forming part of the answer, should be kept apart from it, and crossed out.

5. When a numerical result has to be obtained, see that your calculations are made with the necessary degree of accuracy, but **do not give more significant figures than the nature of the case warrants.** For most results in elementary Mechanics, three or four significant figures is quite accurate enough, and for some cases two such figures will suffice.

6. When your work is completed, **again read through the question and compare it with your answer** to make sure that you have given exactly what is required, and in the form specified.

7. Make sure that your numerical results are reasonable. There is never any excuse for handing in *absurd* results. Commonsense will frequently suffice to tell you whether your figures are or are not reasonable: if not, then make a rough calculation using the nearest **round figures** and so obtain an approximate result to serve as a check on your precise calculations.

8. Do not forget to **count the dimensions** where the reasonableness of a formula or equation is in question.

9. Keep a particularly sharp watch against the following snares: (*a*) **Inconsistent units;** to avoid these, work in systematic units throughout, and convert to practical units, if necessary, at the close. (*b*) **Misplaced decimal points;** use a slide-rule but be careful about the position of the decimal point. (*c*) **Terms dropped out or distorted** during the process of manipulation; see paragraph (4) above.

EXAMPLES I

Attention is drawn to the Hints given above.

1. *A motor-car is moving at a speed of* 42 *miles per hour. Express its speed* (*a*) *in feet per second, and* (*b*) *in centimetres per second.*

$$\begin{aligned}(a)\ \text{Speed of car} &= 42 \text{ miles per hour}\\ &= (42 \times 5{,}280) \text{ feet per hour}\\ &= \frac{42 \times 5{,}280}{60 \times 60} \text{ feet per second}\\ &= 61{\cdot}6 \text{ feet per second.}\end{aligned}$$

$$\begin{aligned}(b)\ \text{Speed of car} &= 61{\cdot}6 \text{ feet per second}\\ &= (61{\cdot}6 \times 30{\cdot}48) \text{ centimetres per second}\\ &= 1{,}877 \text{ centimetres per second.}\end{aligned}$$

(since one foot is equal to 30·48 centimetres).

NOTE.—It is convenient to remember that 15 miles per hour is equal to 22 feet per second. Therefore, to convert miles per hour into feet per second, all we have to do is to multiply by 22 and divide by 15: to convert feet per second into miles per hour, we must multiply by 15 and divide by 22. For example:—

$$\begin{aligned}61{\cdot}6 \text{ feet per second} &= 61{\cdot}6 \times \frac{15}{22} \text{ miles per hour}\\ &= 42 \text{ miles per hour.}\end{aligned}$$

2. Express a speed of 37 miles per hour in F.P.S. units.

3. The mass of a block of wood 2 feet in length, 18 inches in width, and 9 inches in thickness, is one hundredweight. What is its density in F.P.S. units ?

4. A railway train is travelling at a speed of 20 miles per hour, and a motor-car is travelling at a speed of 600 yards per minute. Which is moving the faster ? What is the difference between their speeds, expressed in systematic units ?

5. The density of cast-iron is 450 pounds per cubic-foot, and the density of water is 62·4 pounds per cubic-foot. What is the specific gravity of cast-iron ?

6. The time occupied in doing a certain piece of work is $3\frac{1}{2}$ hours : express this in C.G.S. units of time.

7. The mass of water passing over a water-fall is 2,250 tons per hour. How can this be expressed in F.P.S. units ? If the density of water is 62·4 pounds per cubic-foot, what is the volume passing per minute ?

8. How many C.G.S. units of area are equal to one F.P.S. unit of area ?

9. Calculate the number of cubic-centimetres in one cubic-foot.

10. A man walks at the rate of $3\frac{1}{4}$ miles per hour for an hour and a half. Find how far he has travelled, in systematic units.

11. A train goes from London to Nottingham, a distance of $126\frac{1}{2}$ miles, in 2 hours 27 minutes. What is its average speed (*a*) in practical units, and (*b*) in systematic units ?

12. The mass of a fly-wheel is 4·72 tons, and its volume is 36,800 cubic-inches. Determine the density of the material of which the fly-wheel is composed.

13. What are the dimensions of speed ?

14. Show that the C.G.S. unit of density is 62·4 times as great as the corresponding F.P.S. unit.

15. A furnace consumes half a ton of coal per hour. What is the consumption in pounds per second ?

16. The mass of a steel plate is stated as 20·4 pounds per square-foot of surface. If the density of the material is 489·6 pounds per cubic-foot, what is the thickness of the plate ?

17. A fly-wheel consists of a solid disc of cast-iron, 4 feet in diameter and 6 inches in thickness. Calculate its mass, given that the density of cast-iron is 451 pounds per cubic-foot.

18. A locomotive passes through a railway station at a speed of

43·6 miles per hour. If the platform is 700 feet in length, how long will the engine take to travel from one end of the station to the other?

19. A room used as a store contains 32 tons of tools and apparatus. If the room is 22 feet in length and $14\frac{1}{2}$ feet in breadth, what is the average load per unit area of floor surface?

20. State the dimensions of the following quantities: (*a*) mass per unit length; (*b*) distance travelled per unit time; (*c*) density; (*d*) volume; (*e*) mass multiplied by speed.

21. A train starting at noon from London reaches Exeter at 3.15 p.m. having travelled at an average speed of 78·4 feet per second. What is the distance from London to Exeter?

22. In three successive minutes a motor-car travels distances of 523 yards, 567 yards, and 604 yards. Calculate its average speed (*a*) in miles per hour, and (*b*) in feet per second.

23. What is the C.G.S. equivalent of $63\frac{1}{4}$ miles per hour?

24. If we multiply a density by an area and divide the result by a mass, what is the nature of the quantity which we obtain?

25. Convert a speed of 584 centimetres per second into miles per hour.

CHAPTER 2: FORCES

MECHANICS being concerned with Forces and their effects, we must commence our course of study by considering the nature of force, and some of its properties.

By a **force** in Mechanics we mean what in everyday language we term a **push** or a **pull.**

This is an example of something which occurs very frequently in Mechanics, namely the use of a familiar term in a special restricted sense. In all such cases we have to take particular care that, when we employ such a term, we shall do so only in its limited mechanical meaning and not in its ordinary wider sense.

We shall understand, then, that when we speak of a force we always mean a push or a pull. If we want a formal **definition of force** we can say that it is *any cause which produces, or tends to produce, motion or change of motion in the matter on which it acts.* A little thought will show that the two definitions agree pretty closely: the one is more obvious and the other more exact.

Now, in considering any force, there are two of its properties which we notice at once. One is its **magnitude** or size; the other is that it acts in some definite **direction.** Some forces are great and some are small; some act in one direction and some in another; but they all possess these two properties.

If now we draw a straight line, such that its direction is the same as the direction of a given force, and its length is in some definite proportion to the magnitude of the force, then we have in the straight line a complete **graphical representation** of the force. A force greater than the given force would be represented by a longer line, and a smaller force by a shorter line. Similarly, a force acting in a different direction would be represented by a line in that different direction.

VECTOR AND SCALAR QUANTITIES. We shall see later that forces are not the only quantities which possess these two properties of magnitude and direction: there are a number of others. Any such quantity is termed a **vector quantity.** Many other quantities which occur in Mechanics have magnitude but not direction. These are termed **scalar quantities.** For example, a period of *time* has magnitude but not direction, and is therefore a scalar quantity. Other examples of scalar quantities are *volume* and *density*.

We can now go a step further and state that *all* the quantities with which we have to deal in Mechanics are either vector or scalar quantities. Every such quantity must belong to one group or the other, for the difference between them is simply that vector quantities include the idea of direction, whereas scalar quantities do not. It is worth noticing that the simpler vector quantities, with which we shall chiefly be concerned for the present, have each *one dimension of length* and no more. A recollection of this may serve as a useful check, but it must be noted it is not an invariable rule, so that the dimensions of a quantity do not afford a complete test as to which group it belongs.

We have seen that every force possesses magnitude and direction: there is an additional piece of information that we require before we can say that we know all about the force. We must know *where* it acts on a body. It is usual to state this as the **point of application** of the force: but, for many purposes, a force may be considered to act on a body at *any point* in its own **line of action,** so that it is usually sufficient if we know its line of action, without knowing the actual point at which it is applied.

To make this quite clear, let us consider two equal horizontal forces acting upon a rigid body, and represented by the straight lines AB and CD as shown in Fig. 1. The only difference between the two forces is in their lines of action: therefore, in order to distinguish between

them, it is necessary to know the **location** of each force, as well as its magnitude and direction.

Now consider the force represented by AB. Its line of action intersects the body from E to G. The force may therefore be considered to act on the body at E, or at G, or at any intermediate point F, in its line of action. This is sometimes referred to as the **Principle of the Transmissibility of Force.**

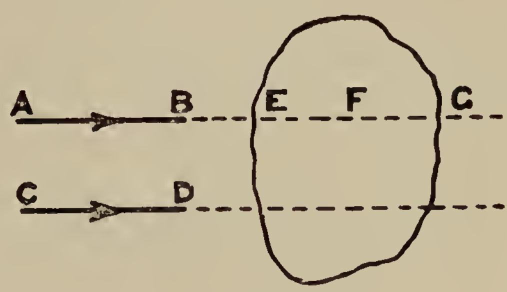


FIG. 1.

EQUILIBRIUM. Let us now consider what happens when a force acts upon a body. If the force is not opposed by any other force, then it will cause the body to move, and the direction of motion will be the same as the direction of the force. If, in spite of the force acting upon it, the body does not move, then we know that the force must be opposed by some other force or forces. An engine pushing a truck on a railway causes it to move: but if another engine were to push at the other end of the truck with equal force, then the truck would remain at rest, because the force exerted by the one engine would be balanced by that exerted by the other in the opposite direction. Similarly, in a "tug-of-war," if the two teams pull equally hard on the rope it will not move in either direction, no matter how great the forces exerted by the men: the forces in one direction are neutralised by those in the other.

In any such case, where a body is acted upon by forces and yet remains at rest, we know that the effects of the forces are counteracting each other. We then say that the body is **in equilibrium** under the action of the forces.

If we have a number of forces acting on a body and we want to know whether it will remain in equilibrium, that is to say, whether it will not be moved by those forces, then we must find some way of determining what

will be the combined effect of the forces. Let us take a very simple case first. **Two equal forces** acting upon a body at the same point, but in exactly opposite directions, will always balance each other. It does not matter at all what their magnitudes may be ; if they are equal and opposite and in the same straight line, the tendency of the one force to move the body in one direction will always be countered by the tendency of the other force to move it in the opposite direction.

It follows from this that the equilibrium of a body will not be disturbed in any way, if, at any point, we introduce two equal and opposite forces ; for the effect of the one will be neutralised by that of the other. Similarly, if from the forces acting upon a body we remove two equal and opposite forces acting at the same point, we shall not by so doing disturb the equilibrium of the body. These results sometimes enable us to simplify our work, so they should be noted.

RESULTANT FORCE. Now suppose that we have two forces acting at the same point on a body, in opposite directions, and that this time the forces are of *unequal* magnitudes. In this case, *part* of the greater force will suffice to balance the smaller force, and the remainder of the greater force will still be available to cause motion. Obviously, the magnitude of this available force will be equal to the arithmetical difference between the magnitudes of the two forces actually acting upon the body. We term this available force, which results from the action of more than one force on a body, the **resultant** of the original forces.

Putting it in a slightly different way, we may say that *the resultant of a number of forces is that force which would have the same effect as the combined effects of all the original forces together.* The resultant, then, of a number of forces acting in the same straight line is equal to the algebraic sum of the forces : that is, to find the resultant, we call the forces acting in one direction **positive,** and those in the opposite direction **negative,** and add

them together in accordance with the rules of algebra.

If, now, we have to deal with a number of forces *not* all in the same straight line, we cannot find the resultant quite so simply, for we must take account of the various directions in which the forces act. Let us consider an illustration.

Suppose that a round table has to be moved across a room, and that two men, P and Q, push it. If they push equally hard but in opposite directions, as shown in Fig. 2 (*a*), then they are wasting their time and energy, for the table will not move. In the language of Mechanics, the forces exerted balance each other, their resultant is zero, and the table is in equilibrium. Now if the men push side by side in the same direction, as shown in Fig. 2 (*b*), the resultant force will be the *sum* of the forces exerted by the men, and, if this resultant is great enough, the table will move in the same direction as the men are pushing.

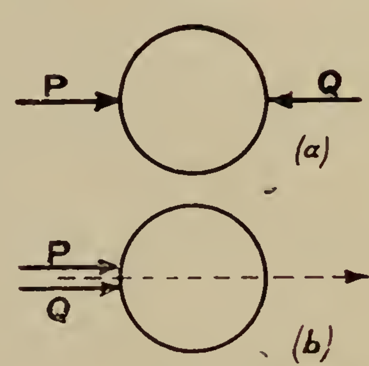


Fig. 2.

Either of these cases is extremely simple, but if now the two men push in *different, but not opposite,* directions, the case is a little more complicated. Experience tells us, however, that the table will move, not in either of the directions in which it is being pushed by the men separately, but in some intermediate direction, as shown in Fig. 3 by the dotted line.

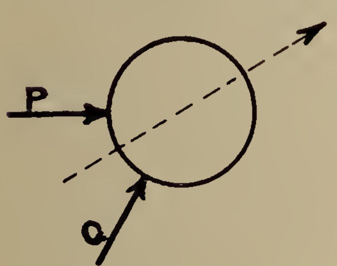


Fig. 3.

THE PARALLELOGRAM LAW. This brings us to one of the fundamental differences between vector and scalar quantities; that is the difference in the methods by which we add together two quantities of the same kind. In the case of scalar quantities we proceed by the ordinary rules of algebra, but in the case of vector quantities we can only do this if the directions of all the quantities are in the same straight line. Usually this

will not be so, and we shall have different directions to take into account as well as magnitudes. In order to add together vector quantities, then, we must have some special method: we find this in what we term the **Parallelogram Law,** which is of extreme importance and should be thoroughly mastered.

This principle may be stated as follows: *If two vector quantities, acting at a point, are represented in magnitude and direction by the two sides of a parallelogram drawn from one point, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram drawn from the same point.*

The actual graphical construction is as follows: Draw, from some point O, a straight line OA, to represent both in magnitude and direction one of the given vector quantities. Such a line is what we term a **vector,** that is, it is the graphical representation of a quantity which possesses both magnitude and direction.

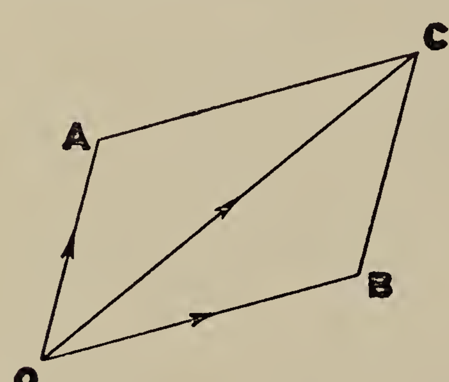


FIG. 4.

Now draw the other vector OB, through the same point O, taking care that the vectors are drawn either *both from* or *both towards* the point O: on no account must one be drawn from and the other towards the point O. Complete the parallelogram by drawing AC through A parallel to OB, and BC through B parallel to OA. Draw the diagonal OC. Then OC represents the resultant of the quantities represented by OA and OB.

Notice that the arrow-heads, which denote what is called the **sense** of each vector, are either all from O or else all towards O. We must take special care to adhere to this rule, as neglect of it will make our results completely wrong.

In practice we can slightly simplify this construction. Since OACB is a parallelogram, AC is equal and parallel to OB. Therefore the quantity which is represented

by the vector OB, could be equally well represented by AC. All we have to do, then, is to draw OA to represent in magnitude and direction the first vector quantity, and AC from A to represent the other vector quantity. OC will then represent the resultant of the quantities which are represented by OA and AC. As before, care must be taken that each vector is drawn in the correct *sense*.

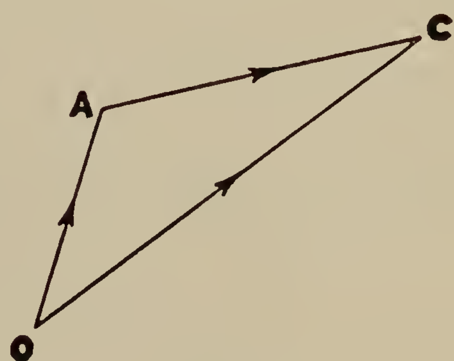


FIG. 5.

This, then, is the method we must employ when we wish to find the resultant of two forces. If we have more than two forces to combine, or **compound** as it is termed, we first take any two of them and find their resultant. We then compound this resultant with a third force to obtain a new resultant, and repeat the process until we have dealt with all the forces. The final resultant so obtained will be the same in whatever order the forces are taken, and will be the **vector sum** of all the original forces.

A mathematical proof of the parallelogram law will be found in the Appendix. This should be studied after Chapter 7 has been mastered, and attention should then be concentrated on understanding it rather than on memorising it. An experimental proof of the Parallelogram of Forces is one of the most usual elementary laboratory exercises, and this should be carefully and thoughtfully performed, while studying this chapter, by every student who has the opportunity.

RESOLUTION OF FORCES. The parallelogram law not only gives us a means of compounding together two or more forces, but also enables us to split up a force into two parts by reversing the process. In this case our problem is usually to find two forces, in two given directions, whose combined effects would be the same as the effect of the original force. Any two forces which can replace a given force in this way, are called the **components** of the given force in the given directions.

Suppose that we have a force represented in magnitude and direction by the vector OA, and wish to replace it by two forces, one in a direction making an angle α with the direction of the given force, and the other in a direction making an angle β with it. The construction is as follows: Draw OB through O at the given angle α with OA, and OC through O at the given angle β with OA. Complete the parallelogram by drawing AB through A parallel to OC, and AC through A parallel to OB. Then OB and OC are the required components in the given directions: that is to say, we can replace the force represented by OA by the two forces represented by OB and OC and they will have exactly the same effect at the point O as the original force.

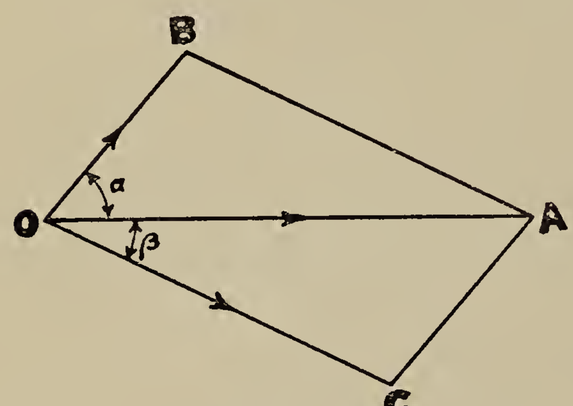


Fig. 6.

Here, again, we can simplify the construction by drawing only the lines OA, OB, and AB. The latter is equal and parallel to OC, so that the required components will now be represented by OB and BA, as shown in Fig. 7.

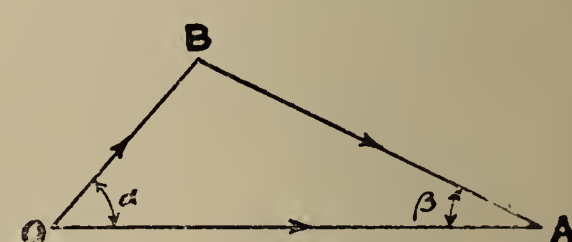


Fig. 7.

Adding forces together is known as the **composition of forces**, and the reverse process of splitting them up into components is known as the **resolution of forces.** Both of these uses of the parallelogram law are of great importance, for much of the more advanced work in Mechanics depends upon them. The student should work numerical examples until he is quite familiar with the methods and understands them thoroughly: he should remember, too, that the law applies to *all* vector quantities.

ACTION AND REACTION. Forces are sometimes classified as either *active* or *passive*. The distinction is

useful in aiding us to think clearly. **Active forces** are those which can originate motion : **passive forces** are those which cannot do so. For example, a man leaning against a wall exerts force upon it : the wall, in resisting the pressure, exerts an equal and opposite force upon the man. The force exerted by the man is of the active kind ; if the opposing force, exerted by the wall, were removed motion would ensue. On the other hand, the force exerted by the wall is of the passive type : if the force exerted by the man were removed the opposing force would disappear, and no motion would take place.

Passive forces, then, are merely called into play by those of the active kind ; they have no independent existence. An active force is sometimes called an **action** and a passive force completely balancing it is then known as a **reaction.** The relation between the two is of the simplest character : *action and reaction are equal and opposite.* This rule, which a little thought will show to be almost axiomatic, is generally called **Newton's Third Law of Motion.** We shall have more to say about it later. It should be noted that while the term *reaction* is in common use in the above sense, *action* is seldom used in this way except in stating Newton's Third Law. There are some passive forces, such as *friction,* which are not reactions in the usual sense.

GRAVITY. There is one particular force which is of very frequent occurrence in Mechanics and is of the greatest importance. That is the force of attraction exerted by the earth on all bodies of every kind. This force is what we usually term **weight** : it may be compared with the force of attraction exerted by a magnet on articles of iron and steel. *The weight of a body is simply the force with which the earth pulls that body towards itself.* Another name for it is the force of **gravity.**

Like every other force, gravity has both magnitude and direction. The magnitude of the force of gravity depends on the distance from the centre of the earth,

so that it is nearly constant for all parts of the earth's surface. The direction is towards the centre of the earth, or *vertical*. To be strictly correct we should say that there are slight variations both of magnitude and of direction. For example, the force of gravity is slightly greater at London than it is at Paris. Also its direction is not in all cases strictly vertical. For ordinary purposes, however, we assume it to be a constant vertical force.

The force with which gravity acts upon a particular body depends also upon how much *matter*, that is, substance or material, there is in the body. The quantity of matter in a body is called its **mass** : this is a very important quantity and one with regard to which it is especially necessary to be quite clear.

Suppose that we have some solid body, such, for example, as a cricket-ball. Its mass is constant wherever we may take it : if we could go to the centre of the earth its mass would be unchanged, for it depends solely upon the amount of material in the body. The **weight** of the ball is quite a different affair, for, as we have already seen, the weight of a body is the *force* with which the earth attracts it, and that force depends on the distance from the centre of the earth. Therefore, if we could pay our imaginary visit to the centre of the earth, we should find that the cricket-ball had lost all its weight, although there was no change in the ball itself.

It is necessary to emphasize this point, because, in spite of the very obvious differences between them, *mass* and *weight* are frequently confused. Before passing on we must make certain that we fully appreciate the distinction.

We can now go a step further and state that, *at any given place*, the weight of a body is directly proportional to its mass. If we have two bodies, of which the mass of one is twice the mass of the other, then the weight of the one will also be twice the weight of the other, provided that the weights are both measured at the same place. If we now take both bodies to some other

place, their weights may be slightly changed in actual magnitude, but will still be in the same *ratio* one to the other. The masses will be entirely unchanged.

We see therefore, that :—

MASS is the quantity of matter in a body : WEIGHT is the force with which the earth attracts the body.

MASS is constant everywhere : WEIGHT varies in different places.

MEASUREMENT OF FORCE. We have seen that force may be defined as any cause which produces, or tends to produce, motion or change of motion in the matter upon which it acts. This property of force can be utilised as a measure of its magnitude, for the greater the force the greater will be its effect in producing motion. The standard or unit force will be that force which produces unit change of motion in unit time. If we employ **systematic units** in measuring the change of motion produced and the time occupied, then the force will be measured in systematic units also.

The systematic unit of force in the F.P.S. system is known as the **poundal.** The systematic unit of force in the C.G.S. system is known as the **dyne.** Further consideration of these units must be deferred until we have become acquainted with some of the properties of motion itself : for the present we will simply note their names, and remind ourselves that, as they are *systematic,* they must also be *absolute.*

The other method of measuring the magnitude of a force is to compare it with the magnitude of the force exerted by gravity upon a particular mass. In this country we most often choose for this purpose a mass of one pound : the force with which gravity acts on this mass is what we term *the weight of a pound,* or **pound-weight.** Using the metric system we may similarly employ the *weight of a gramme,* or **gramme-weight** as a unit of force. We may, of course, extend the idea and use the weight of *any* mass as a unit of force ; the weight of a ton, for example.

If, then, we have a force which is of exactly the same magnitude as the force of gravity acting on a mass of one pound, we say that we have a force of *one pound-weight.* A force twice as great will be termed a force of *two pounds-weight,* and so on. Notice that the forces we measure in this way are not necessarily in the same *direction* as the force of gravity : it is the magnitudes which are equal. The pound-weight, the gramme-weight, and all similar units are called **gravitation units** of force : they are *variable* units—since gravity is a variable force—and are therefore unsuitable for scientific work.

We can now appreciate better the necessity for getting very clear ideas on the distinction between mass and weight, if confusion is to be avoided in our work. It will be an aid to accuracy if we make it a definite rule never to employ the terms *pound* or *gramme* for forces, reserving them solely for their proper use as measures of mass. For forces the correct names of the gravitation units are the *pound-weight* and the *gramme-weight.*

We must defer the consideration of the *dimensions* of force and weight until a later chapter. We may, however, note that the dimensions of a mass are *one dimension of mass,* and are denoted therefore by **[M].**

SUMMARY OF CHAPTER 2

A **force** is any cause which produces, or tends to produce, motion or change of motion in the matter upon which it acts.

Every force has **magnitude, direction,** and **point of application,** and it can be completely represented by a straight line. Such a straight line is known as a **vector**.

All physical quantities are either *vector* or *scalar* quantities. **Vector quantities** have both magnitude and direction, while **scalar quantities** have magnitude only.

When a body is acted upon by forces and yet is not moved by them, it is said to be **in equilibrium**.

The **resultant** of a number of forces is that force which would have the same effect as all the original forces together.

Vector quantities must be added together by means of the **Parallelogram Law.** Adding forces together in this way is known as the

composition of forces, and the reverse process of splitting them up into **components** as the **resolution of forces.**

Forces are either **active** or **passive**: the former can originate motion; the latter cannot. Passive forces of a particular type are frequently termed **reactions.** Newton's **Third Law of Motion** states that *action and reaction are equal and opposite.*

Gravity or **weight** is the force with which the earth attracts all bodies. **Mass** is the quantity of matter in a body. *Mass* and *weight* are entirely different quantities and must on no account be confused.

The *systematic* units of force are the **poundal** (F.P.S. system) and the **dyne** (C.G.S. system). The corresponding *gravitation* units are the **pound-weight** and the **gramme-weight.** The systematic units are, of course, *absolute*: the gravitation units are *variable.*

EXAMPLES II

(For Hints on Working Examples, see page 21.)

1. *Find the resultant of two forces, one of* 19 *poundals, and the other of* 23 *poundals, with an angle of* 39 *degrees between their lines of action.*

(*a*) **Graphical Solution.** Choosing a suitable scale, we draw the vector AB to represent the force of 23 poundals. Next we set

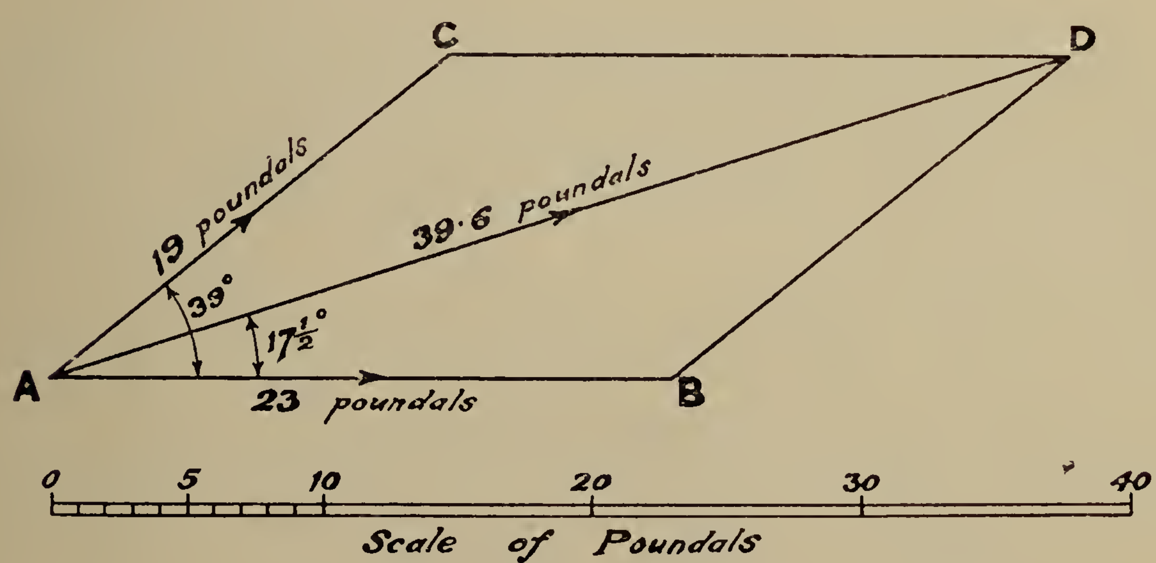


FIG. 8.

off the angle BAC = 39°, and along AC we measure off a length to represent the force of 19 poundals, to the same scale as before.

Drawing CD parallel to AB, and BD parallel to AC, to meet at D, we complete the parallelogram. Then we draw the diagonal AD, and this represents, to the same scale, the resultant of the two forces.

Measuring the diagonal AD we find that the resultant of the two

given forces is a force of 39·6 poundals, in a direction making an angle of $17\frac{1}{2}$ degrees with the direction of the force of 23 poundals.

(*b*) **Solution by Trigonometry.**

$$\begin{aligned}\text{Resultant force} &= \sqrt{(AC)^2 + (AB)^2 + 2(AB \times AC \times \cos 39^\circ)} \\ &= \sqrt{(19)^2 + (23)^2 + 2(19 \times 23 \times \cdot 7772)} \\ &= \sqrt{361 + 529 + 679} \\ &= \sqrt{1{,}569} \\ &= 39\cdot 6 \text{ poundals.}\end{aligned}$$

Tangent of angle between forces AD and AB

$$\begin{aligned}&= \frac{AC \sin 39^\circ}{AB + AC \cos 39^\circ} \\ &= \frac{19 \times \cdot 6293}{23 + (19 \times \cdot 7772)} \\ &= \frac{19 \times \cdot 6293}{37\cdot 75} \\ &= \cdot 3166 \\ &= \tan 17\tfrac{1}{2}^\circ.\end{aligned}$$

2. *Two men move a table across a room, exerting forces equal to* 12 *and* 17 *pounds-weight respectively, in directions which are at right angles to each other. Afterwards another man moves the table back to its original position. Find the direction and magnitude of the force which he exerts.*

(*a*) **Graphical Solution.**

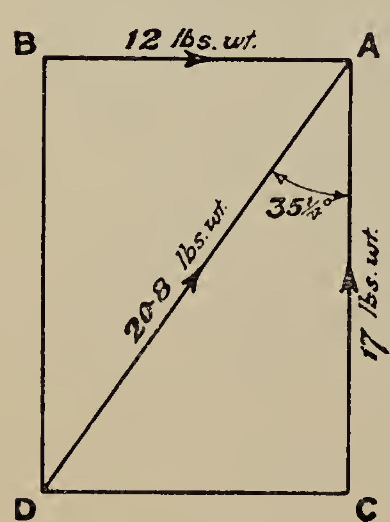


Fig. 9.

The pushes exerted by the two men are (by the parallelogram of forces) together equal to a push of 20·8 pounds-weight in a direction making an angle of $35\frac{1}{4}$ degrees with the direction of the 17 pound-weight push.

Therefore the force exerted by the third man, in returning the table to its original position, must be equal and opposite to this resultant, i.e., if the forces exerted by the two men in the first case are represented by BA and CA on the diagram, the force exerted by the third man is represented by AD on the diagram.

(*b*) **Solution by Trigonometry.**

Resultant push exerted by the two men

$$= \sqrt{(12)^2 + (17)^2}$$
$$= \sqrt{144 + 289}$$
$$= \sqrt{433}$$
$$= 20 \cdot 8 \text{ pounds-weight.}$$

This must therefore be the magnitude of the push exerted by the third man when returning the table to its original position.

Tangent of angle between direction of this force and the force of 17 pounds-weight

$$= \frac{12}{17}$$
$$= \cdot 7055$$
$$= \tan 35\tfrac{1}{4}^\circ$$

3. *What are the components of a vertical force of* 42 *pounds-weight in directions making angles of* 23 *degrees and* 51 *degrees with the vertical ?*

(*a*) **Graphical Solution.**

First we draw the vector AB, representing the force of 42 pounds-weight. Then from A we set off lines AC and AD, at the given angles, and through B draw lines BC and BD, to complete the parallelogram.

Then AC and AD represent, to scale, the required components of the given force.

Component of given force at angle of 23 degrees with the vertical = 33·9 pounds-weight.

Component of given force at angle of 51 degrees with the vertical = 17·2 pounds-weight.

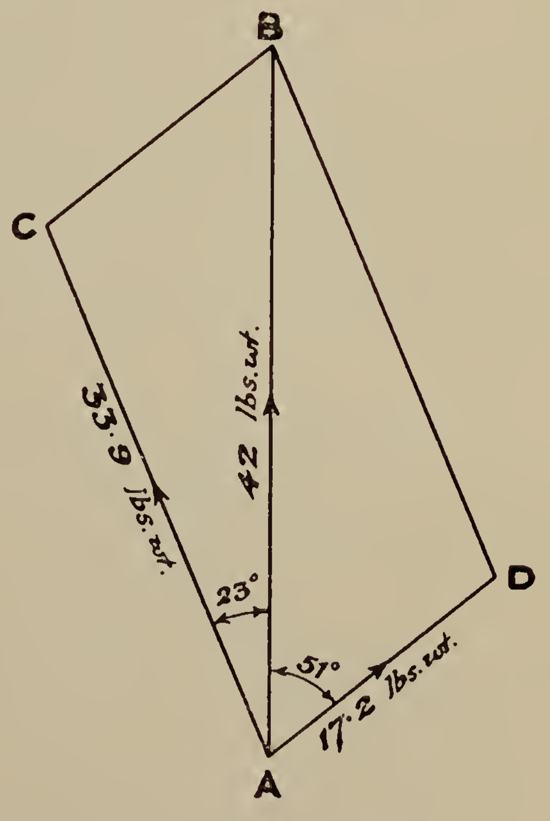


FIG. 10.

(*b*) **Solution by Trigonometry.**

Since the sides of any triangle are proportional to the sines of the opposite angles :

$$\therefore \frac{AD}{\sin 23^\circ} = \frac{42 \text{ pounds-weight}}{\sin 106^\circ}$$

whence $$AD = \frac{42 \times \cdot 3907}{\cdot 9613} \text{ pounds-weight}$$
$$= 17 \cdot 2 \text{ pounds-weight.}$$

Also $$\frac{AC}{\sin 51^\circ} = \frac{42 \text{ pounds-weight}}{\sin 106^\circ}$$

whence $$AC = \frac{42 \times \cdot 7772}{\cdot 9613} \text{ pounds-weight}$$
$$= 33 \cdot 9 \text{ pounds-weight.}$$

4. *Find the vertical and horizontal components of a force of* 73 *poundals acting at an angle of* 30 *degrees from the horizontal.*

Vertical component of force

$$= 73 \text{ poundals} \times \sin 30^\circ$$
$$= (73 \times \tfrac{1}{2}) \text{ poundals}$$
$$= 36 \cdot 5 \text{ poundals.}$$

Horizontal component of force

$$= 73 \text{ poundals} \times \cos 30^\circ$$
$$= (73 \times \cdot 866) \text{ poundals}$$
$$= 63 \cdot 2 \text{ poundals.}$$

The same results will be obtained by drawing the parallelogram of forces, as shown, and measuring the lines AB and BC representing the required components.

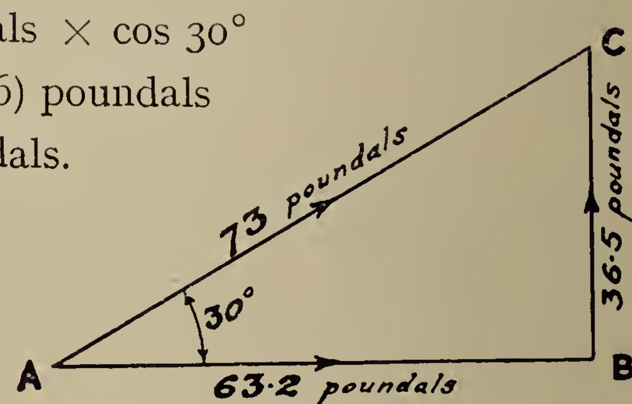


Fig. 11.

5. A body is acted upon by a horizontal force of 27 poundals from left to right, and an upward force of 33 poundals at an angle of 63 degrees from the direction of the first force. Find the resultant force on the body.

6. Find the resultant of two forces, one of $5\frac{1}{4}$ tons-weight and the other $2\frac{3}{4}$ tons-weight, with an angle of 100 degrees between their lines of action.

7. Find the force which will have the same effect as a force of 32 pounds-weight and a force of 23 pounds-weight with an angle of 84 degrees between them.

8. Three forces, P, Q, and R, act at the same point and in the same plane. P is a vertical downward force of 105 poundals, Q is a force of 67 poundals upwards at 30 degrees from the vertical, and R

is a force of 80 poundals upwards at 70 degrees from Q. Find their resultant.

9. Compound together a horizontal force of 360 poundals and a vertical force of 272 poundals.

10. Find the horizontal and vertical components of a force of 18 pounds-weight inclined at an angle of 30 degrees from the horizontal.

11. Resolve a vertical force of 12 tons-weight into components making angles of 40 degrees and 36 degrees with the vertical.

12. A man weighing 10 stone sits on a table. What are the forces acting between the man and the table? Give their magnitudes in pounds-weight, and their directions, and state whether they are active or passive.

13. Find the resultant of a force of $12\frac{1}{2}$ pounds-weight and a force of 33 pounds-weight if the angle between their lines of action is 45 degrees.

14. A force P, of magnitude 50 poundals, acts at an angle of 37 degrees from the horizontal upwards, and a force Q of magnitude 64 poundals, acts upwards at an angle of 100 degrees from P. Find the sum of their resolved parts (*a*) in a horizontal direction, and (*b*) in a vertical direction.

15. A force P of 254 poundals acts at an angle of 33 degrees from the horizontal. Find the force which has the same horizontal component as P, and a vertical component twice as great as that of P.

16. Find the resultant of two forces each of 56 pounds-weight with an angle of 55 degrees between them.

17. Find the horizontal and vertical components, in pounds-weight, of a force of ·45 ton-weight acting at an angle of 20 degrees from the vertical.

18. Resolve a horizontal force of 320 poundals into components at angles of 52 degrees and 47 degrees from the horizontal.

19. Find the vertical and horizontal resolved parts of a force of 73 pounds-weight acting at an angle of 41 degrees from the horizontal.

20. A force P is of magnitude 88 poundals and acts at an angle of 120 degrees from the vertical. A force Q has its vertical component equal to twice the vertical component of P, and its horizontal component equal to half the horizontal component of P. Find the magnitude and direction of Q and the resultant of P and Q.

21. Compound together three forces, P, Q, and R, which act outwards from the same point O. The magnitudes of the forces are 7 pounds-weight, 8 pounds-weight, and 10 pounds-weight, respec-

tively, and their lines of action are given by the position of the minute-hand of a clock at five minutes past the hour, 25 minutes past the hour, and 20 minutes to the hour, respectively.

22. Determine the sum of the components (*a*) horizontal, and (*b*) vertical, of the following forces:

(i) A force of 28 pounds-weight upwards at 30 degrees from the vertical.

(ii) A force of 21 pounds-weight at 180 degrees from the first force.

(iii) A force downwards of 18 pounds-weight midway between the other forces.

23. Find the sum of the resolved parts of the following forces, (*a*) horizontal, and (*b*) vertical:

(i) A force of 97 poundals upwards to the right at an angle of 65 degrees from the horizontal.

(ii) A force of 132 poundals upwards to the left at an angle of 57 degrees from the horizontal.

(iii) A force vertically downwards of 224 poundals.

(iv) A force of 86 poundals downwards to the right at an angle of 10 degrees from the horizontal.

CHAPTER 3: MOMENTS

WE have seen that in addition to the magnitude and direction of a force, we require to know either its point of application or its line of action. The effect which a force will have on a body depends considerably upon *where* it is applied.

Let us suppose that a body is fixed in such a way that the only movement of which it is capable is one of rotation, that is, it is hinged at some point or axis about which it is free to turn; for example, a door, which can turn about its hinges. Then if a force is applied to the body, of sufficient magnitude and in a suitable direction, not through the hinge, motion will actually take place and the body will rotate about the fixed point.

The tendency of the force to cause the body to turn about the fixed point or axis depends on two factors: one is the *magnitude* of the force, and the other is the *perpendicular distance of its line of action* from the fixed point or axis. The greater is either of these quantities, the greater will be the tendency of the force to turn the body.

We can then, in any particular case, obtain whatever **turning effect** may be required, in a great variety of ways. If we employ a small force, then we must apply it at a greater distance from the fixed point or axis than if we employed a large force. If we reduce the distance, then we must increase the force. To obtain the same turning effect we must keep the product *force* $\times$ *perpendicular distance* constant.

For example, the amount of force required to turn a door on its hinges depends on where the force is applied. The nearer to the hinges we push at it, the greater the effort required. Again, if we were to push at the edge of the door, straight towards the hinges, no amount of force would move the door, because the line of action

of the push passes through the hinge, and therefore its perpendicular distance from the hinge is zero. To turn the door most easily, that is, with the smallest effort, we apply our force as far away as possible from the hinges, and perpendicularly to the surface of the door.

The tendency of a force to turn a body about a fixed point or axis is called the **torque** or **turning moment,** or simply the **moment** of the force about the point or axis. The perpendicular distance of the line of action of the force from the fixed point or axis is termed the **arm** of the force about that point or axis.

MEASUREMENT OF TURNING MOMENT. The magnitude of the moment is measured by the product of the force and its perpendicular distance from the axis, i.e.,

moment = force × arm.

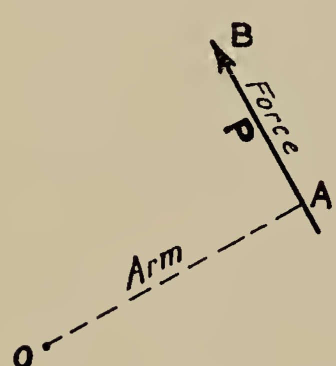


FIG. 12.

In Fig. 12, if O be the fixed point, and AB the line of action of a force P, then the turning moment of P about O is :—

$$\mathbf{P \times OA}$$

where OA is perpendicular to AB.

If the force is measured in *pounds-weight* and the arm in *feet*, then the turning moment will be measured in **pound-feet**. It may be noted that the correct term is **poundsweight-feet**, but it is customary to abbreviate this to pound-feet. If the force is measured in *poundals* and the arm in *feet*, then the turning moment will be measured in **poundal-feet.** Similarly, turning moments in the C.G.S. system will be measured in **gramme-centimetres** if we employ gravitation units of force, and in **dyne-centimetres** if we employ absolute units of force.

POSITIVE AND NEGATIVE MOMENTS. It will readily be seen that a force may tend to turn a body in either of two different ways. It may tend to turn it in the same way as the hands of a clock ; or in the

opposite direction. To denote these two different kinds of moment we use the terms *positive* and *negative*.

A **positive turning moment** is one which tends to turn a body in a counter-clockwise direction, and a **negative turning moment** is one which tends to turn it in a clockwise direction.

Positive ↺ Negative ↻

GENERAL DEFINITION OF A MOMENT. We may here note that the term **moment** does not apply only to forces, although it is sometimes used as though that were its only application. We can extend the idea to *any* physical quantity, and determine the moment of the quantity about some given axis. We might not be able to find any use for some of the moments which we could obtain in this way, but a number of them are of considerable importance. In each case we find the moment of the quantity by multiplying it by its effective distance from the given axis. Such a moment may be more precisely specified by calling it the **first moment** of the quantity about the given axis.

We can obtain another class of moment by multiplying a quantity by the *square* of its effective distance from the given axis. Such a moment is termed the **second moment** of the quantity about the given axis.

We are now in a position to define moments in general terms, and may do so as follows:

The **first moment** *of any quantity about any axis is equal to the product of the quantity and its effective distance from the given axis.*

The **second moment** *of any quantity about any axis is equal to the product of the quantity and the* **square** *of its effective distance from the given axis.*

As examples of the use of moments of other quantities than force, we may mention the following cases:—

(*a*) The *first moment of a mass*: this may be used in determining the position of the centre of mass of a body.

(*b*) The *first moment of an area*: this may be used in

determining the position of the centre of area or centroid of an area.

(*c*) The *second moment of a mass*: this is usually known as the **moment of inertia** of a body, and occurs very frequently in dealing with rotating bodies.

(*d*) The *second moment of an area*: this is sometimes wrongly termed the moment of inertia of an area. It is of great importance in the theory of structures.

(*e*) The *first moment of momentum*, usually called simply the moment of momentum, or the **angular momentum** of a body.

We shall deal with many of these moments in later chapters.

RESULTANT TURNING MOMENT. If several forces act upon a body and tend to turn it about a given axis, then the total turning effect which they will exert upon it will be the algebraic sum of the separate moments.

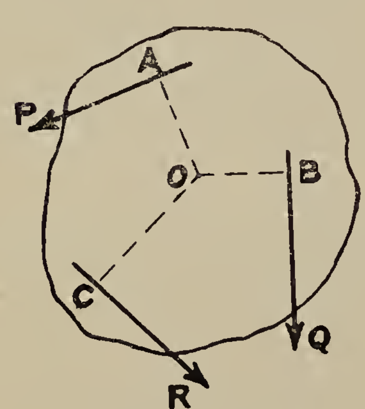


Fig. 13.

Suppose that forces P, Q, and R act upon a body as shown in Fig. 13. Then their turning moments about the point O are respectively :—

$$+(P \times OA)$$
$$-(Q \times OB)$$
and $$+(R \times OC).$$

The algebraic sum of the moments is therefore :—

$$(P \times OA) - (Q \times OB) + (R \times OC).$$

It can easily be shown that the **resultant moment** found by adding algebraically the separate moments in this way, is equal to the moment, about the given axis, of the resultant of the several *forces*. This proposition is usually known as **Varignon's Theorem.** A formal proof will be found in the Appendix.

PRINCIPLE OF MOMENTS. If several forces act in one plane upon a rigid body, each force will separately tend to turn the body about any fixed point in

the plane (unless the line of action of the force passes through the fixed point). Corresponding, therefore, to the several *forces* we shall have a number of *moments*, some of them positive or counterclockwise, and some of them negative or clockwise. If the body remains **in equilibrium** under the action of the forces, then the *resultant moment* of the forces about the fixed point must be zero, that is, the sum of the positive moments must be equal to the sum of the negative moments.

Clearly this is true about whatever point in the plane we take moments. This result, which is known as the **Principle of Moments,** may therefore be stated in the following way. *If a body is in equilibrium under the action of forces in one plane, then the sum of the clockwise moments of the forces about any point in the plane must equal the sum of the counterclockwise moments about the same point.*

This is one of the most useful and important of the principles of Mechanics. It is also one of the easiest to understand and to apply. It should therefore be mastered quickly and thoroughly.

Let us look at the question from a slightly different point of view. If a body is at rest although it is under the action of a number of forces, then we know that these forces have no *resultant* tendency to turn the body, as otherwise it would not remain at rest. It follows necessarily that the sum of the moments of the forces which tend to turn the body in a clockwise direction, must balance the sum of the moments of the forces which tend to turn it in the opposite direction.

Suppose, for example, that a number of men are pushing at the arms of a capstan. So long as they all push in the same direction, say clockwise, so long will the capstan rotate and carry out its duties. If, however, some of the men push in one direction and some in the contrary direction, then the motion of the capstan

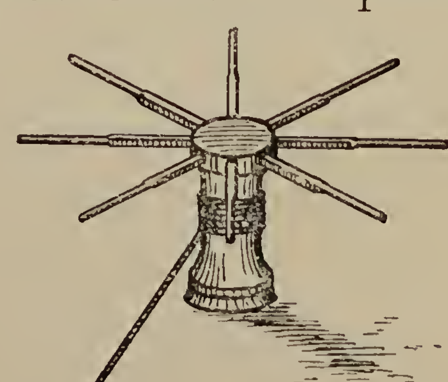


FIG. 14.

will depend on which party exerts the greater turning moment. If the two parties are equally strong and equally determined, then their efforts will neutralise each other and the capstan will not turn at all. The only good that they will do is to provide us with an illustration of the Principle of Moments!

LEVERS. One of the commonest practical applications of the Principle of Moments is to be found in the **lever.** This, in its simplest form, is a straight bar supported at some point about which it is free to turn. This point of support is termed the **fulcrum** of the lever.

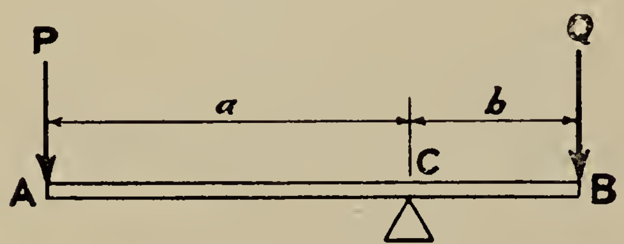


FIG. 15.

Let AB (Fig. 15) represent a lever having its fulcrum at C. Suppose that forces P and Q are applied to the ends A and B of the lever, at distances of a and b respectively from the fulcrum. The *moments* of these two forces about the fulcrum will oppose each other, and if they balance (that is, if the lever is in equilibrium) then we have:—

$$P \times a = Q \times b,$$

whence $P = \frac{b}{a}.Q$ and $\frac{P}{Q} = \frac{b}{a}.$

That is to say, P is to Q as b is to a.

Now let us suppose that the force Q is supplied by the weight of a body placed on the end of the lever at B. Then if a is greater than b as shown in the figure, the force P required to balance the weight Q will be less than Q in the ratio b/a. When the forces are arranged in this way, so that their moments about the fulcrum just balance, then the lever is in equilibrium and no motion takes place.

If we now slightly increase the force P, it will cause the end of the bar on which it acts to move downwards, and the other end of the bar to rise, carrying with it the body placed thereon.

Therefore, by means of the lever, a small force P can be made to overcome a large force Q. In any particular case, if we wish to overcome a resisting force Q by means of a smaller force P, all that we have to do is to use a lever having the distances from the fulcrum so arranged that :—

$$b/a = P/Q.$$

In any such case the resisting force Q is termed the **load** and the force P, which moves it, is called the **effort.**

The arrangement shown diagrammatically in Fig. 15 is actually employed in the **crowbar** (see Fig. 16), in the juvenile "**see-saw,**" in various **weighing machines,** and in many other practical machines and appliances. The student should see how many different examples of the lever he can find in use, not merely in the laboratory or workshop, but also in domestic and general affairs.

FIG. 16.

Such an investigation will quickly show that the relative positions of the *fulcrum,* the *load,* and the *effort* are not always as we have shown them above. Both effort and load may be on the same side of the fulcrum, instead of on opposite sides. They must then, of course, act in opposite directions. Levers so arranged may be divided into two groups, according to whether the load or the effort is nearer to the fulcrum.

It is customary, therefore, to arrange levers in three classes according to the relative positions of the fulcrum, effort, and load. In the first class we place levers which have the fulcrum between the load and the effort. In the second and third classes we place those levers which have the load and the effort both on the same side of the fulcrum : in the former the *load* is nearer to the

fulcrum ; in the latter the *effort* is nearer. Each of these arrangements is illustrated diagrammatically in Fig. 17. (I), (II), and (III) show levers of the first, second, and third classes respectively.

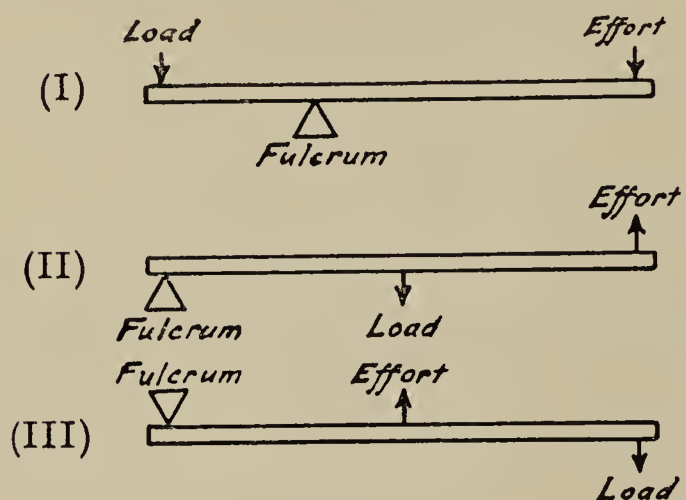


Fig. 17.

We can most easily recall this classification if we remember which of the three forces acting on the lever is in the middle (the reaction of the fulcrum is, of course, the third force). We have :—

Class I. **Fulcrum**
Class II. **Load**
Class III. **Effort**
} between the other two forces.

PARALLEL FORCES. In the previous chapter we saw how to find the resultant of a number of forces by means of the *Parallelogram Law.* There is, however, one case in which this construction fails.

If we have to find the resultant of a number of forces acting in one plane upon a rigid body, and the forces are all parallel, then we cannot draw any parallelogram with two of its adjacent sides representing two of the forces, for none of the forces will intersect anywhere but at infinity, and so our rule breaks down.

We can, however, easily obtain the resultant by means of the Principle of Moments. If we have a system of parallel forces, the sum of the moments of the forces about any point in their plane must be just equal to the moment of their resultant about the same point (by Varignon's Theorem). By finding the sum of the moments of the forces, therefore, we can determine the moment of the resultant, and if we can also find the *magnitude* of the resultant, then we can quickly find its line of action. Alternatively, if we have found the moment of the resultant about a given point, and we

also know its line of action, then we can readily determine its magnitude.

It is clear that if a system of parallel forces is in equilibrium, then the sum of the forces acting in one direction must be equal numerically to the sum of the forces acting in the opposite direction. Therefore the magnitude of the resultant of a system of parallel forces must be numerically equal to the *difference* between the sum of the forces in one direction, and the sum of the forces in the opposite direction.

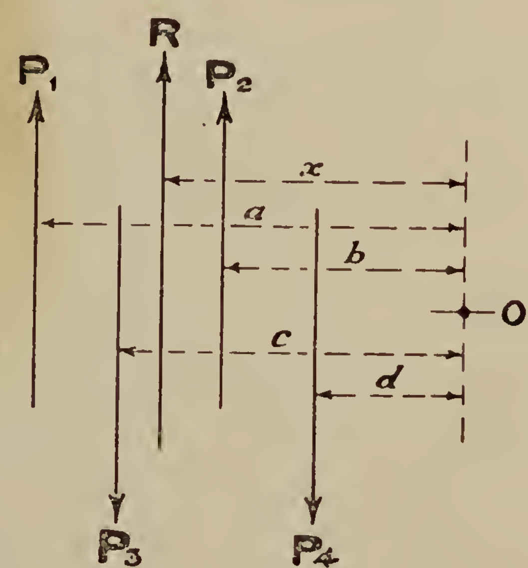


FIG. 18.

Let P_1, P_2, P_3, P_4 . . . . . be parallel forces acting in one plane as shown in Fig. 18. Then the sum of the magnitudes of the forces acting in one direction is $P_1 + P_2$ . . . and the sum of the magnitudes of the forces acting in the opposite direction is $P_3 + P_4$ . . . . . Therefore the magnitude of the resultant of the system of forces is :—

$$R = (P_1 + P_2 \ldots) - (P_3 + P_4 \ldots)$$
$$= P_1 + P_2 - P_3 - P_4 \ldots\ldots$$

Let O be any point in the plane of the forces, and let a, b, c, d, ... be the perpendicular distances of the lines of action of the forces from O, and x the perpendicular distance of the line of action of the resultant R from O.

Then, taking moments about O, we have :—

$$R \times x = (P_1 \times a) + (P_2 \times b) - (P_3 \times c) - (P_4 \times d)\ldots.$$

$$\text{whence } x = \frac{P_1.a + P_2.b - P_3.c - P_4.d\ldots\ldots.}{P_1 + P_2 - P_3 - P_4 \ldots\ldots}$$

A practical application of this method is employed very frequently in connection with loaded beams. Usually in such a case we require to find the vertical

reactions R_1 and R_2 at the points A and B where the beam is supported (see Fig. 19), due to a series of vertical loads, P_1, P_2, P_3, etc. Clearly, the beam must be in equilibrium, and the sum of the reactions R_1 and R_2 must therefore be equal to the sum of the loads P_1, P_2, P_3... Also, taking moments about B, the moment of R_1 about B must be equal and opposite to the sum of the moments of the loads P_1, P_2, P_3 ..... about the same point. Hence we can calculate the magnitude of R_1 and therefore the magnitude of R_2 which will be equal to the difference between the sum of the loads, and the reaction R_1, i.e., it will be equal to $(P_1 + P_2 + P_3 \ldots) - R_1$.

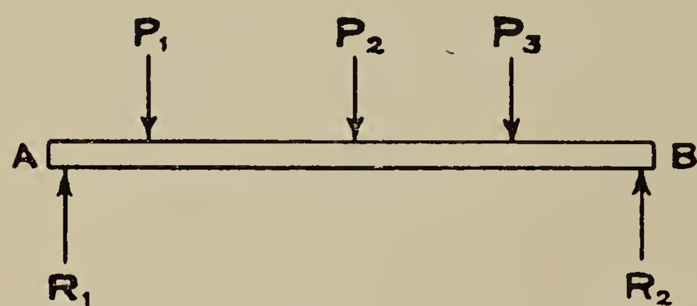


FIG. 19.

By this method we can, in general, find the resultant of any number of parallel forces, but there is one particular case in which it fails.

COUPLES. If we have two forces equal in magnitude, opposite in direction, and not in the same straight line, then by the ordinary rules, the magnitude of their resultant would be zero. It is, however, quite obvious that these forces are not in equilibrium, for if we take moments about any point in their plane, the resultant *moment* is not zero, as (by the Principle of Moments) it would be if the forces were in equilibrium.

Such a pair of forces, which are *equal, parallel,* and *unlike,* is termed a **couple.** The perpendicular distance between the lines of action of the forces is called the **arm** of the couple, and the product of *one* of the forces and the arm of the couple gives us the **moment of the couple.** If each of the forces is of magnitude P, and the arm of the couple is a distance *a,* then the moment of the couple is P.*a* (see Fig. 20). The moment of a couple is measured in the same units as we employ for the moment of a force, i.e., for turning-moment,

and the dimensions of the two quantities are the same.

Since, as we have seen, no single force is the resultant of a couple, so also it is impossible to find any single force which is the **equilibrant** of a couple. In other words, *it is impossible to find any single force which will balance a couple.*

We may say that a couple is a system of forces which has a zero resultant, but whose turning moment is *not* zero, so that a couple simply tends to cause **rotation** of the body upon which it acts, without any motion of **translation**, that is, without any motion from one place to another.

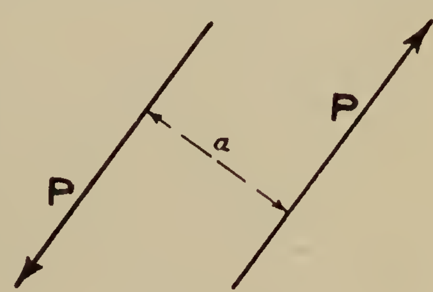


FIG. 20.

To balance such a system of forces we must have another system of forces, whose moment must be equal to the moment of the first system but in the opposite direction, and whose resultant must also be zero. Clearly such a system of forces constitutes another couple. Therefore we may say that *a couple can only be balanced by another couple of equal and opposite moment.*

Now, as we have already seen, the moment of any couple is equal to the product of one of the forces and the perpendicular distance between their lines of action. Therefore any couple of moment **P.a** can be balanced by any couple whose moment **Q.b** is equal and opposite to P.a. So long, therefore, as the *product* Q.b is equal and opposite to the *product* P.a, we can vary the actual values of the force Q and the distance b in any way we choose. Evidently, *any* two couples in the same plane whose moments are equal and opposite will balance one another.

The most important property of couples in general is that *the moment of a couple is the same about any point in its plane, and is equal to the moment P.a of the couple.*

Let O be any point in the plane of the couple, P the magnitude of each of the forces composing it, a the perpendicular distance between the two forces, and b

the perpendicular distance of the point O from the nearer force P, as shown diagrammatically in Fig. 21.

Taking moments about O we have :—

$$\text{Resultant moment} = P(a + b) - P.b$$
$$= P.a + P.b - P.b$$
$$= P.a$$ which is the moment of the couple.

This is true for *any* point O in the plane of the couple, so the proposition is proved. This special property of a couple should be carefully studied and remembered, as it is very useful. We see that a couple, unlike a force, has no particular position or point of application, but may be shifted to anywhere in its plane without altering its effect.

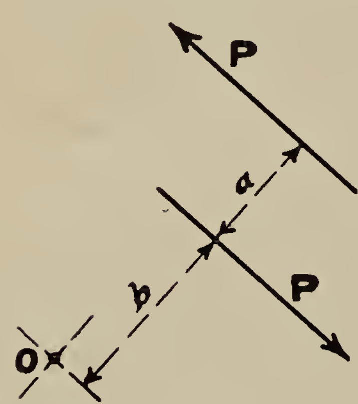


Fig. 21.

Since the whole effect of a couple is measured by its turning moment, and is unaffected by its position, it follows that *a couple may be replaced by any other couple in the same plane and of equal moment.*

It follows also that *any number of couples in the same plane, acting upon a rigid body, are equivalent to any single couple, of which the moment is equal to the algebraic sum of their moments.*

It can be shown, also, that a couple may be shifted to any *parallel* plane without altering its effect upon a rigid body. Consequently *any two couples in parallel planes whose moments are equal and opposite will balance one another.*

TRANSFERRED FORCE. A proposition which is sometimes very useful is as follows: *a single force, acting at any point, is equivalent to the same force acting at any other point, plus a couple of which the moment is equal to the product of the force and the perpendicular distance between its two positions.*

Let P_1 (Fig. 22) be any force acting at any point A.

At any other point B we may introduce two equal and opposite forces P_2 and P_3, each of the same magnitude as P_1 and parallel to it. The introduction of these two forces will (as shown in Chapter 2, page 28) not in any way disturb the existing state of affairs, since they are equal and opposite and so balance each other.

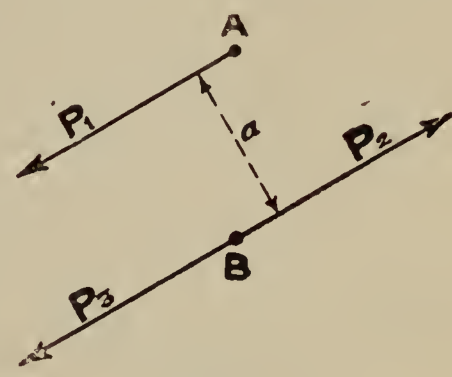


FIG. 22.

Now P_3 is equal to the original force and acts at B. Also P_1 and P_2 together constitute a couple whose moment is equal to $P_1.a$, where a is the perpendicular distance between the lines of action of P_1 and P_3. The proposition is therefore proved.

EXAMPLES OF COUPLES. Whenever a body is caused to rotate there must be a couple acting upon it, and the student should notice any instance which he may encounter of a body being set in rotation, and should reason out for himself what are the forces constituting the couple in that particular case.

The *capstan,* to which reference was made earlier in the chapter, provides such an instance, for the forces exerted by the men in setting it in motion produce rotation and not translation, and therefore reduce to a couple. When a door is caused to turn upon its hinges, the couple is formed by the force exerted by the hand upon the door, and the equal and opposite *reaction* from the hinges on to the door.

SUMMARY OF CHAPTER 3

The tendency of a force to turn a body about a fixed point or axis is termed the **torque** or **turning moment** of the force about the point or axis.

Turning moment = Force × Arm.

Turning moments are measured in **pound-feet** or **gramme-centimetres** if we use gravitation units of force, and in **poundal-feet** or **dyne-centimetres** if we use absolute units of force.

Positive turning moments are those which tend to turn a body

in a counterclockwise direction: **negative turning moments** are those which tend to turn it in a clockwise direction. The *resultant moment* of a number of forces acting upon a rigid body, about a given point or axis, is equal to the algebraic sum of the moments of the separate forces about the same point or axis.

Varignon's Theorem states that the resultant moment of a number of forces about a given point is equal to the moment of the resultant force about the same point.

The **first moment** of any quantity about any axis is equal to the product of the quantity and its effective distance from the given axis.

The **second moment** of any quantity about any axis is equal to the product of the quantity and the **square** of its effective distance from the given axis.

The **Principle of Moments** states that if a body is in equilibrium under the action of forces in one plane, then the sum of the clockwise moments of the forces about any point in the plane must be equal to the sum of the counterclockwise moments of the forces about the same point.

The **lever** is an application of the Principle of Moments. Levers are acted upon by three forces, viz., the **effort**, the **load**, and the **reaction of the fulcrum**, and are classified according to which of these forces occupies the middle position between the other two. Class I: Fulcrum. Class II: Load. Class III: Effort.

The resultant of a system of **parallel forces** can be obtained by means of the Principle of Moments, except in the case of two equal, unlike, parallel forces.

Two equal, unlike, parallel forces constitute a **couple.** The moment of a couple is equal to the product of one of the forces and the perpendicular distance between them, and is constant about any point in the plane of the couple.

A couple has zero resultant but its moment is not zero. It cannot, therefore, be balanced by any single force, but only by another couple of equal and opposite moment.

A couple may be replaced by any other couple of equal moment in the same or any parallel plane.

Any number of couples in the same plane, acting upon a rigid body, are equivalent to any single couple, of which the moment is equal to the algebraic sum of their moments.

A single force acting at any point is equivalent to the same force acting at any other point, plus a couple of which the moment is equal to the product of the force and the perpendicular distance between its two positions.

EXAMPLES III

(*For Hints on Working Examples, see page* 21.)

1. *Find the moments about a point, of the following forces :—(i) A force of* 23 *poundals at a perpendicular distance of* 7 *feet from the point. (ii) A force of* 9·2 *pounds-weight at a perpendicular distance of* 3 *feet* 4 *inches from the point. (iii) A force of* $2\frac{1}{4}$ *tons-weight at a perpendicular distance of* $1\frac{1}{2}$ *inches from the point.*

(i) Moment of given force about point

= force × arm
= 23 poundals × 7 feet
= 161 poundal-feet.

(ii) Moment of given force about point

= force × arm
= 9·2 pounds-weight × $3\frac{1}{3}$ feet
= 30·67 pound-feet.

(iii) Moment of given force about point

= force × arm
$= (2\frac{1}{4} \times 2{,}240)$ pounds-weight $\times \frac{1\frac{1}{2}}{12}$ foot
= 630 pound-feet.

2. *ABCD is a square of* 3 *feet side. A force of* 7 *poundals acts along AB, a force of* 12 *poundals along CB, a force of* 8 *poundals along CD, and a force of* 10 *poundals along DA. Find the resultant moment (i) about A, and (ii) about the centre of the square.*

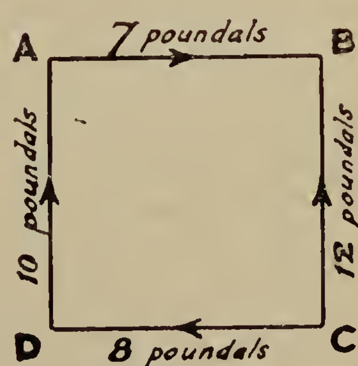


FIG. 23.

Resultant moment about A

= (7 pdls. × 0 ft.) + (12 pdls. × 3 ft.) − (8 pdls. × 3 ft.) + (10 pdls. × 0 ft.)
= (0 + 36 − 24 + 0) poundal-feet
= 12 poundal-feet.

Resultant moment about centre of square

$= -(7 \text{ pdls.} \times 1\frac{1}{2} \text{ ft.}) + (12 \text{ pdls.} \times 1\frac{1}{2} \text{ ft.}) - (8 \text{ pdls.} \times 1\frac{1}{2} \text{ ft.}) - (10 \text{ pdls.} \times 1\frac{1}{2} \text{ ft.})$
$= \{(-7 + 12 - 8 - 10) \times 1\frac{1}{2}\}$ poundal-feet
$= -(13 \times 1\frac{1}{2})$ poundal-feet
$= -19\frac{1}{2}$ poundal-feet.

3. *A beam, 12 feet in length, is supported at each end and carries the following loads: $\frac{1}{2}$ ton at 2 feet from one end, 2 tons at the centre, and $1\frac{3}{4}$ tons at 2 feet 6 inches from the other end. Find the reactions at the supports.*

In solving any problem of this kind, the first thing to do is to make a sketch and mark on it all the available information. (*See notes on page* 21).

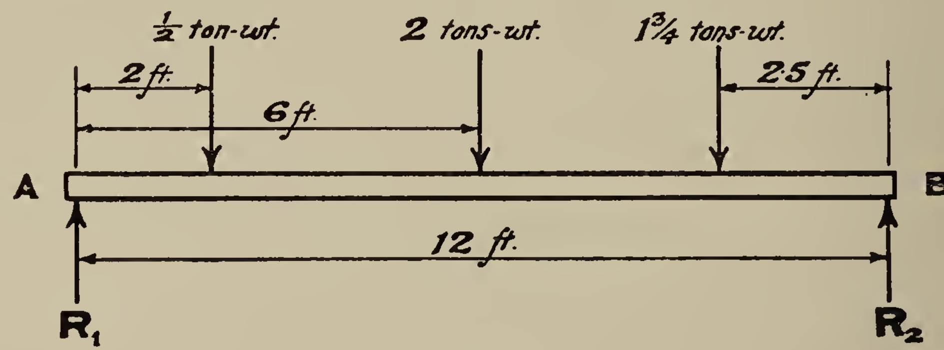


FIG. 24.

Taking moments about A:—

$$R_2 \times 12 \text{ feet} = (\tfrac{1}{2} \text{ ton-wt.} \times 2 \text{ ft.}) + (2 \text{ tons-wt.} \times 6 \text{ ft.}) + (1\tfrac{3}{4} \text{ tons-wt.} \times 9{\cdot}5 \text{ ft.})$$
$$= 1 \text{ ton-foot} + 12 \text{ ton-feet} + 16{\cdot}63 \text{ ton-feet}$$
$$= 29{\cdot}63 \text{ ton-feet,}$$

whence
$$R_2 = \frac{29{\cdot}63 \text{ ton-feet}}{12 \text{ feet}}$$
$$= 2{\cdot}47 \text{ tons-weight.}$$

Therefore
$$R_1 = (\tfrac{1}{2} + 2 + 1\tfrac{3}{4} - 2{\cdot}47) \text{ tons-weight}$$
$$= (4{\cdot}25 - 2{\cdot}47)$$
$$= 1{\cdot}78 \text{ tons-weight.}$$

4. *A uniform rod AB, of 6 pounds-weight and 8 feet in length, is suspended from a beam by vertical cords attached to the rod at A and a point 2 feet from B respectively. A load of 7 pounds-weight is hung from the rod 3 feet from A, and a load of 3 pounds-weight at B. Find the tensions in the cords.*

Let T_1 = tension in cord at A.
T_2 = tension in other cord.

Taking moments about A:—

$$T_2 \times 6 \text{ feet} = (3 \text{ lbs. wt.} \times 8 \text{ ft.}) + (6 \text{ lbs. wt.} \times 4 \text{ ft.}) + (7 \text{ lbs. wt.} \times 3 \text{ ft.})$$
$$= (24 + 24 + 21) \text{ pound-feet.}$$
$$= 69 \text{ pound-feet.}$$

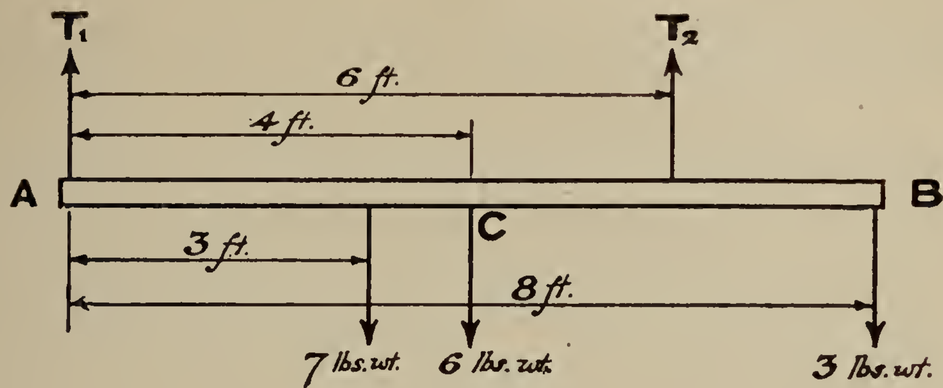


FIG. 25.

$$\text{whence } T_2 = \frac{69 \text{ pound-feet}}{6 \text{ feet}}$$
$$= 11\tfrac{1}{2} \text{ pounds-weight.}$$

Therefore

$$T_1 = (7 + 6 + 3) \text{ pounds-weight} - T_2 \text{ pounds-weight}$$
$$= (16 - 11\tfrac{1}{2}) \text{ pounds-weight}$$
$$= 4\tfrac{1}{2} \text{ pounds-weight.}$$

(*Note.*—The weight of the rod itself may be considered to act at its middle point C, as shown in the figure.)

5. *A beam AB,* 40 *feet in length, is supported at A and at a point* 10 *feet from B. It carries an evenly distributed load of* 2 *cwt. per foot run, and concentrated loads of* 3 *and* 5 *tons-weight at points* 5 *feet from A and B respectively. Find the reactions at the supports.*

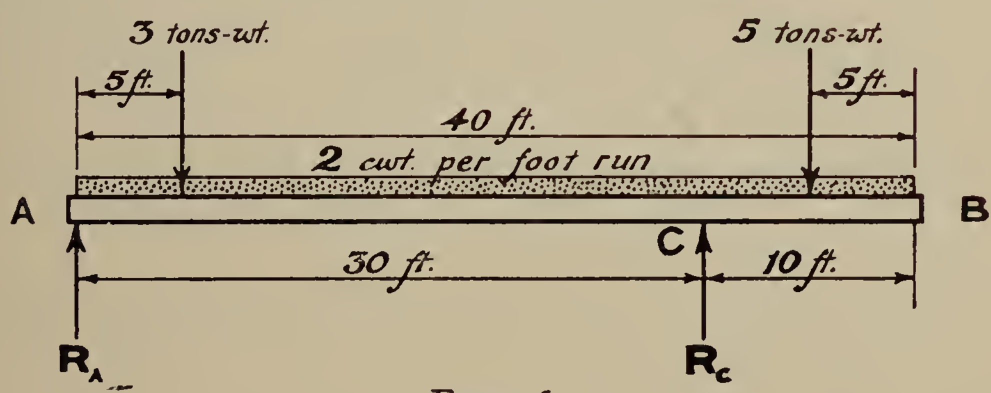


FIG. 26.

The evenly distributed load of 2 cwt. per foot run on a span of 40 feet is, for the purpose of finding the reactions, equivalent to a load of (2 cwt. × 40), i.e., to a load of 4 tons-weight acting at the middle of the length AB.

Taking moments about A :—

$$R_C \times 30 \text{ feet} = (3 \text{ tons-wt.} \times 5 \text{ ft.}) + (4 \text{ tons-wt.} \times 20 \text{ ft.}) + (5 \text{ tons-wt.} \times 35 \text{ ft.})$$
$$= (15 + 80 + 175) \text{ ton-feet}$$
$$= 270 \text{ ton-feet.}$$

$$\therefore R_C = \frac{270 \text{ ton-feet}}{30 \text{ feet}}$$
$$= 9 \text{ tons-weight,}$$

so that $R_A = (3 + 4 + 5 - 9)$ tons-weight
$= 3$ tons-weight.

These results may be checked by taking moments about the other point of support, C,

$$R_A \times 30 \text{ feet} = (3 \text{ tons-wt.} \times 25 \text{ ft.}) + (4 \text{ tons-wt.} \times 10 \text{ ft.}) - (5 \text{ tons-wt.} \times 5 \text{ ft.})$$
$$= (75 + 40 - 25) \text{ ton-feet}$$
$$= 90 \text{ ton-feet.}$$
$$\therefore R_A = \frac{90 \text{ ton-feet}}{30 \text{ feet}}$$
$$= 3 \text{ tons-weight, as before,}$$

so that $R_C = (3 + 4 + 5 - 3)$ tons-weight
$= 9$ tons-weight, as before.

6. Find the moment of a force of 57 pounds-weight about a point so placed that the perpendicular distance of the line of action of the force from the point is 14 feet.

7. Find the moments, about a point, of the following forces :— (*a*) A force of 234 poundals at an effective distance of 8 inches from the point. (*b*) A force of $1\frac{3}{4}$ tons-weight at an effective distance of 2·4 inches from the point. (*c*) A force of 98 pounds-weight at an effective distance of 3 feet 6 inches from the point.

8. What is the turning moment of a force of 112 pounds-weight acting with an arm of 30 inches ?

9. Find the second moment of mass (moment of inertia) of a body of 50 pounds mass, about an axis at an effective distance of 8 feet from the body.

10. Which force will have the greater turning effect upon a body, a force of 83 pounds-weight at a perpendicular distance of 35 feet, or a force of 1·3 tons-weight at a perpendicular distance of $14\frac{1}{2}$ inches ? If these forces act in opposite directions, what resultant turning effect will they have upon the body ?

11. What force must be applied to a body at an effective distance of 2 feet $7\frac{1}{4}$ inches from a fixed axis, in order to balance a turning moment of 38·7 pound-feet about that axis ?

12. Find the first moment of an area of $1\frac{1}{2}$ square-feet about an axis at a mean distance of 24·9 feet from it.

13. The mass of a fly-wheel is $4\frac{1}{2}$ tons, and its mean effective diameter is 5 feet 10 inches. Calculate its moment of inertia about its axis of rotation.

14. Determine the dimensions of the following quantities: (i) First moment of mass. (ii) First moment of area. (iii) Second moment of mass (moment of inertia). (iv) Second moment of area.

15. A boy weighing $4\frac{1}{2}$ stone sits on one end of a "see-saw" at a distance from the support of 4 feet 8 inches, and another boy weighing $3\frac{3}{4}$ stone sits on the other end. How far must the second boy be from the support in order that the see-saw may balance? What is the turning moment due to the weight of each boy?

16. A horizontal beam 14 feet in length is supported at each end and carries a central load of 3 tons-weight. What is the reaction at each support?

17. A beam, 17 feet 6 inches in length, supported at the ends, A and B, carries the following loads: 2 tons-weight at 3 feet from A; $5\frac{1}{2}$ tons-weight at the middle point of the beam; and 3 tons-weight at B. Find the reactions at the supports.

18. A steel beam, AB, 32 feet in length, is supported at A and at a point $4\frac{1}{2}$ feet from B. Calculate the reactions at the supports when the beam is loaded in the following manner: a load of 16 tons-weight evenly distributed along the whole length of the beam; a load of 22 tons-weight at a point 18 inches from A; a load of $24\frac{1}{2}$ tons-weight at a distance of 14 feet from A; and a load of 8·3 tons-weight at B.

19. A beam, 15 feet 6 inches in length, is supported at two points each 2 feet 9 inches from the centre of the beam, and carries loads of 210 pounds-weight at each end, and a load of $\frac{1}{2}$ ton-weight at a point one foot from the centre. Find the reactions at the supports.

20. A force of 347 pounds-weight is applied to a lever, at a distance of one yard from the fulcrum. (*a*) What is the torque exerted on the lever, in systematic units? (*b*) What load, in poundals, could be lifted by the lever at a point $14\frac{1}{2}$ inches from the fulcrum?

21. ABCD is a square of 4 feet 6 inches side. Forces act in the following directions: a force of 332 poundals along BA, a force of 400 poundals along BC, a force of 215 poundals along CD, and a force of 280 poundals along DA. Find the resultant moment of the forces about each of the points A, B, C, and D, and about the centre of the square.

22. A beam projects 5 feet from a wall and carries a load of $\frac{3}{4}$ ton-weight at its outer end. Find the bending-moment (i.e., the torque) on the beam at the point of support.

23. Calculate the moment of inertia about a given axis of a body which consists of two parts: the mass of one of the parts is

56 pounds and its effective distance from the axis is 3 feet; the mass of the other part is 37 pounds and its effective distance from the axis is 3 feet 6 inches.

24. Find the resultant torque upon a body, free to turn about an axis, when the following forces act upon it: a force of 455 poundals at a perpendicular distance of 34 inches from the axis; a force of 520 poundals at a perpendicular distance of 2 feet; a force of 116 poundals at a distance of 4 feet 7 inches; and a force of 208 poundals at a distance of 10 inches. The moments of the first and last forces are clockwise: the moments of the others are counterclockwise.

25. A pole 20 feet in length rests horizontally across the top of a wall. The weight of the pole is 154 pounds, and the pole is of uniform cross-section. If 13 feet of the pole projects beyond the wall and carries a load of 56 pounds-weight at the outer end, what will be the tension in a vertical rope holding down the other end of the pole?

26. A beam, 18 feet in length, is supported at one end, A, and at a point B. Loads are placed upon the beam as follows: a load of 88 pounds-weight at A, a load of 43 pounds-weight at 4 feet from A, a load of 24 pounds-weight at 12 feet from A, and a load of 30 pounds-weight at the other end of the beam. If the beam itself weighs $1\frac{1}{4}$ cwt., what is the greatest distance possible between A and B so that the beam may not overbalance, and what is then the reaction of the support at B?

27. Find the moment of inertia (second moment of mass) of a body about a given axis, if the body is composed of the following parts: a mass of 4 pounds at a distance of 7 inches from the given axis, a mass of $12\frac{1}{2}$ pounds at a distance of one foot from the axis, two masses each of $4\frac{1}{4}$ pounds at a distance of 15 inches from the axis, and a mass of 14 ounces at a distance of 3 feet 6 inches from the axis.

CHAPTER 4: EQUILIBRIUM

WE have already dealt with some points in connexion with the equilibrium of bodies under the action of forces. We must now consider the question in a more general way.

Let us first take the case of three forces acting at a point. This we deal with by means of the **Triangle of Forces,** an extremely important law which was first stated by Simon Stevinus, a Dutch mathematician, in the year 1586.

THE TRIANGLE OF FORCES. *If three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, then they are in equilibrium.*

Suppose that P, Q, and R are the three forces acting at the point O, and that they are represented by the sides of the triangle ABC taken in order, AB representing the force P, BC the force Q, and CA the force R, as shown in Fig. 27.

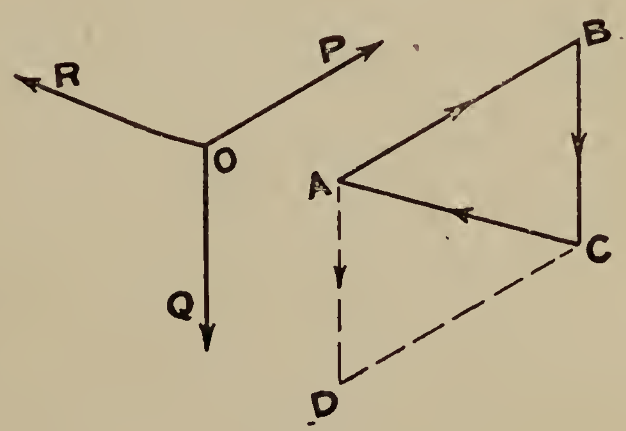


FIG. 27.

Complete the parallelogram ABCD, so that AD is equal and parallel to BC. AD can then be taken as representing the force Q, instead of BC.

Then, by the Parallelogram of Forces, the resultant of the force P (represented by AB) and the force Q (represented by AD) is represented both in magnitude and direction by AC. But CA represents the third force R. The resultant of the forces P and Q is therefore a force equal and opposite to the force R and acting at the same point, so that the three forces must be in equilibrium.

The *converse* of the Triangle of Forces is also true, viz., that if three forces acting at a point are in equilibrium, then they can be represented in magnitude and direction by the sides of a triangle taken in order.

The meaning of the expression "taken in order" will be readily understood if it is noticed that the force P is represented by the vector AB drawn *from* A *to* B, the force Q by the vector BC drawn *from* B *to* C, and the force R by the vector CA drawn *from* C *to* A, so that the arrow-heads, denoting the *sense* of each force, follow the same way round the triangle.

Care must be taken that the two theorems which are known as the *Triangle of Forces* and the *Parallelogram of Forces* are not confused with each other. They somewhat resemble each other and are, in fact, statements of two different aspects of the same principle, but the distinction between them must be noted. The *Parallelogram of Forces* enables us to find the resultant of two forces acting at a point: the *Triangle of Forces* deals with the equilibrium of three forces acting at a point.

RESOLVED PARTS OF A FORCE. We have seen, in Chapter 2, that we can split up any force into a pair of components which will have the same effect as the original force. There is one particular case of this resolution of forces which is of special importance, and that is when the components are in directions which are perpendicular to each other. We then term them the **resolved parts** of the force in the given directions. The distinction between *resolved parts* and *components* should be noticed, because they are sometimes confused.

The directions in which we most frequently require to know the resolved parts of a force are *vertical* and *horizontal*. The process of finding these resolved parts we call **resolving vertically** and **resolving horizontally**.

EQUILIBRIUM OF FORCES AT A POINT. We can now state the conditions of equilibrium for any number of forces *acting at a point*.

If a number of forces acting at a point are in equilibrium, the algebraic sums of their resolved parts in two directions at right angles are separately zero.

We can easily see that this is true, for if the forces are in equilibrium their resultant must be zero; and if their resultant is zero, then the resolved parts of the resultant in any directions must also be zero. But if the resolved part of the resultant in any direction is zero, then the algebraic sum of the resolved parts in that direction of the original forces, must also be zero; for any resultant is simply that force which will have the same effect as the forces which it replaces.

The *converse* of this theorem is also true; that is, that if the sums of the resolved parts, in two directions mutually perpendicular, of a number of forces acting at a point, are separately zero, then the forces are in equilibrium.

THREE FORCES ACTING ON A RIGID BODY. In considering the equilibrium of a rigid body acted upon by three forces in one plane, we shall find the following theorem very useful.

If three forces acting in one plane upon a rigid body are in equilibrium, they must either meet in a point or be parallel.

If the forces are not all parallel, then evidently at least two of them must meet at a point. Suppose that we are dealing with three forces, P, Q, and R, and that P and Q meet at a point O, as shown in Fig. 28. Then, by the Parallelogram of Forces, the resultant of P and Q must also pass through the same point O. Now the resultant of P and Q can only be balanced by a force which is equal and opposite to it and in the same straight line, and which must therefore pass through the same point O. Therefore, if the three forces are in equili-

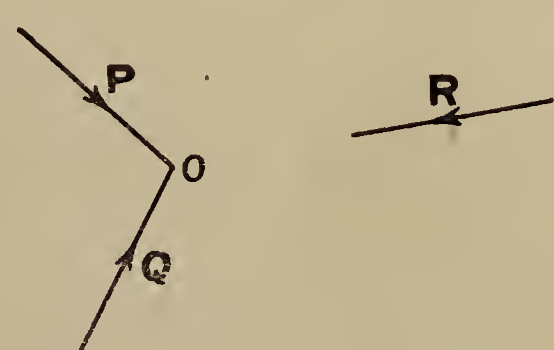


Fig. 28.

brium, the third force R, which has to balance the resultant of the other two forces, must pass through the point O which is the point of intersection of those forces. That is to say, all the three forces must meet in one point.

It should be noticed that, if three forces acting in one plane upon a rigid body keep it in equilibrium, then *no two of the forces can together be less than the third force*: otherwise the resultant of the two forces would also be less than the third force, which is impossible if we have equilibrium.

Another useful theorem is as follows: *If three forces acting upon a rigid body keep it in equilibrium, they must all lie in the same plane.* Let P, Q, and R be the three forces. The resultant of P and Q must be a single force equal and opposite to R and acting in the same straight line: therefore *any* plane containing the resultant of P and Q must contain R. But P and Q must lie in one plane with their resultant: therefore they must lie in one plane with R.

GENERAL CONDITIONS OF EQUILIBRIUM. We can now discuss the conditions of equilibrium for a rigid body which is acted upon by any number of forces, all of them in one plane, but not necessarily meeting at one point. Let us endeavour to reason them out from considerations of ordinary common-sense.

It will help us to do this if we realise that there are *two distinct kinds of motion* which may be given to a body, two different ways in which it may be made to move. It may shift its position from one point to another, moving along a straight line, say from A to B, as shown in Fig. 29; this we call motion of **translation**: or it may turn round and round about some fixed point such as A; this we call motion of **rotation.**

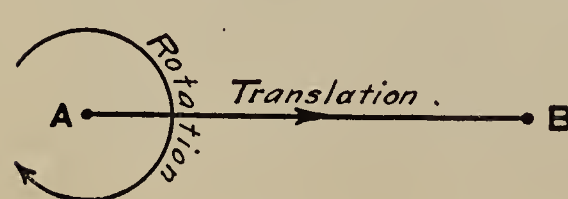


Fig. 29.

A lift, moving from floor to floor of a building, has motion of translation; so has the piston of an engine

as it moves to and fro ; so has a ship sailing along a straight course.

The sails of a windmill have motion of rotation ; so has the fly-wheel of an engine ; so have the hands of a clock.

A body may move with both kinds of motion at the same time. The wheels of a vehicle in motion provide us with a very good example of this. The whole vehicle, including the wheels, moves along with motion of translation, and at the same time the wheels also rotate.

Now if the forces acting on a body are in equilibrium, they will not cause the body to move in *either* of these ways : they will not cause it to rotate, nor will they cause it to change its position from one place to another.

We see, therefore, that the resultant of the forces acting on a body must not cause it to move in *any* way if equilibrium is to be maintained ; for the resultant is simply that force which would have exactly the same effect as the original forces.

Now if the forces acting upon a body reduce to a *single force* in any direction, then they will cause the body to move in that direction : hence for equilibrium the forces must not reduce to a single force in any direction.

Also if the forces acting upon a body reduce to a pair of equal, unlike, parallel forces, that is, to a *couple,* then they will cause the body to rotate : therefore, for equilibrium, the forces must not reduce to a couple.

These two conditions are all that we require : we can put them together and state that, for equilibrium, the forces acting in one plane upon a rigid body, must not reduce to either a single force in any direction, or to a couple. Conversely, *if the forces acting in one plane upon a rigid body, do not reduce to either a single force in any direction or to a couple, then they are in equilibrium.*

These conditions are *necessary* ; for otherwise motion will ensue. They are also *sufficient* ; for if they are fulfilled motion cannot ensue.

EQUILIBRIUM CONDITIONS IN PRACTICE. Although the conditions stated above are both *necessary* and *sufficient,* we find, when we come to apply them in practice, that there is more than one way of determining whether a system of forces does or does not comply with them. There are, in fact, three such ways in regular use: let us consider each in turn, remembering that we are dealing throughout with forces acting in one plane upon a rigid body.

I. *If the algebraic sum of the moments of the forces about each of* **three** *points in the plane, not in the same straight line, is separately zero, then the forces are in equilibrium.*

The forces in this case cannot reduce to a couple, for if they did, then the sum of their moments about any point in the plane would be equal to the moment of the couple, and could not be zero.

Neither can the forces reduce to a single force, for a force must either pass through any given point, or else have a moment about it. As the moments about each of the three points vanish, any resultant force would have to pass through each of the points, and, as they are not in the same straight line, this is obviously impossible.

II. *If the algebraic sum of the moments of the forces about each of* **two** *points, A and B, in the plane is separately zero, and the algebraic sum of the resolved parts of the forces in the* **one** *direction along AB is also zero, then the forces are in equilibrium.*

Obviously, the forces cannot reduce to a couple, for a couple would not have zero moment about any point in its plane. Also, any resultant force, if it has no moment about either A or B, must act along AB: but the resolved parts of the forces in this direction vanish, so that the system cannot reduce to any single force.

III. *If the algebraic sum of the moments of the forces about any* **one** *point in their plane is zero, and the alge-*

braic sum of the resolved parts of the forces in any **two** *directions is separately zero, then the forces are in equilibrium.*

Here, again, the forces cannot reduce to a couple, for they have no resultant moment about the given point. Neither can they reduce to a single force, for they have no resultant in either of two directions : in other words, the resultant force has zero components in two different directions, and is therefore itself zero.

It should be noted that, although it is not necessary that the directions in which the forces are resolved should be at right angles to each other, it is usually best to choose them so.

SOLUTION OF PROBLEMS ON EQUILIBRIUM. In practice the third method given above will be found the most generally suitable. The procedure will then be as follows :—

(*a*) Find the algebraic sum of the resolved parts of the forces in any convenient direction (*e.g.*, vertically), and equate it to zero.

(*b*) Find the algebraic sum of the resolved parts of the forces in a direction at right angles to that previously taken (*e.g.*, horizontally), and equate it to zero.

(*c*) Find the algebraic sum of the moments of the forces about any point in the plane, and equate it to zero.

This gives a series of simultaneous equations from which we can determine the desired results. The method requires modification in particular cases according to the nature of the information given, and the results sought. It will be more readily understood by reference to the worked examples at the end of the chapter.

When solving problems on the equilibrium of forces, it is of great importance to commence by setting down on paper, in the form of a sketch, all the data available. Whether it is intended to solve the question analytically or graphically, such a sketch will be found of the greatest value in enabling the student to see what information

he has at his disposal, and in assisting him, first to choose the most suitable method of attacking the problem, and then to carry out the work with ease and certainty. It is desirable, after completing the sketch, to check it carefully to make sure that *all* the information given has been included, and that no error has been made in putting it down.

Much of the difficulty which is experienced in dealing with questions of equilibrium is due to lack of clear perception of the nature of the particular problem which has to be solved in a given case. If the student does not really know what he is trying to do, he is hardly likely to do it either with ease or success. A regular habit of making neat, accurate sketches before commencing calculations will therefore be found of very great service, since it enables one to visualise the problem to an extent that would otherwise be almost impossible. The object of the problem should be kept in mind the whole time and all work directed definitely to that end. The student who works systematically and is conscious of a deliberate purpose in his efforts is not only likely to achieve better results in the subject of Mechanics, but is also giving himself mental training of inestimable value.

EQUILIBRIUM OF CONSTRAINED BODIES.

In considering the equilibrium of a rigid body acted upon by a system of forces in one plane, we have, so far, assumed that the body was free to move in *any* way. It is, however, quite possible for a body to be constrained so that motion of **translation** is prevented, while the body remains free to receive motion of **rotation.** An example of a body constrained in this way is a door, which can only turn about its hinges.

Clearly, in such a case the conditions of equilibrium are simplified, for no system of forces can cause the body to move from one place to another. Unless, therefore, the forces acting are such as to cause the body to turn about its fixed point or axis, the body will be in equilibrium under their action. In other words, the body is

in equilibrium unless the forces acting upon it reduce to a couple.

There are two slightly different cases of this constraint, for if the body is only constrained at one *point*, then rotation may take place in *any* plane which passes through that point, whereas if the body is constrained at an *axis*, then rotation can only take place in a plane which is perpendicular to that axis. A ball-and-socket joint, such as that joining the arm to the shoulder, gives an approximate illustration of the first, and a hinged lid or door gives an illustration of the second.

The conditions of equilibrium in the two cases may be expressed as follows :—

I. *If a rigid body is fixed at one point, and is acted upon by a number of forces in a plane which passes through the fixed point, it is in equilibrium if the algebraic sum of the moments of the forces about the fixed point is zero.*

II. *If a rigid body is fixed at an axis, and is acted upon by a number of forces, whose directions are perpendicular to the fixed axis, it is in equilibrium if the algebraic sum of the moments of the forces about the fixed axis is zero.*

Suppose that we have a body which is constrained in one of these ways, and that it is acted upon by a system of forces which has no resultant tendency to *turn* the body, but which does, nevertheless, reduce to a single force acting upon the body. How is it that we have equilibrium in such a case, for, at first sight, there would appear to be an unbalanced resultant force acting upon the body?

Obviously, since the resultant of the system of forces acting upon the body exerts no turning moment upon it, the resultant must pass through the fixed point or axis. To balance it, therefore, another force is required, equal, opposite, and in the same straight line. This will be supplied by the **reaction** of the hinge or pivot on to the body.

In every such case, therefore, the system of forces

acting on the body, together with the reaction from the hinge or pivot, either reduce to a couple which will cause the body to rotate about the hinge or pivot, or else they are in equilibrium.

SUMMARY OF CHAPTER 4

The **Triangle of Forces** states that if three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, then they are in equilibrium. The converse is also true.

The components of a force in directions which are mutually perpendicular are termed the **resolved parts** of the force in those directions.

If a number of forces acting at a point are in equilibrium, then the algebraic sums of their resolved parts in two directions at right angles are separately zero.

If three forces acting in one plane upon a rigid body are in equilibrium, they must either meet in a point or be parallel. Also, if three forces acting upon a rigid body keep it in equilibrium, they must all lie in the same plane.

If a system of forces is in equilibrium, it will not cause either **translation** or **rotation** : therefore, for equilibrium, the forces acting upon a body must not reduce to either a single **force** in any direction or to a **couple.**

A system of forces acting in one plane upon a rigid body is in equilibrium if :—

I. The sum of the moments of the forces about each of **3** points in the plane, not in the same straight line, is zero; or :

II. The sum of the moments of the forces about each of **2** points, A and B, in the plane is zero, and the sum of the resolved parts of the forces along AB is also zero; or :

III. The sum of the moments of the forces about any **1** point in their plane is zero, and the sum of the resolved parts of the forces in any **2** directions is zero.

In solving problems on equilibrium we usually :—

(*a*) Find the sum of the resolved parts of the forces in a vertical direction, and equate to zero.

(*b*) Find the sum of the resolved parts of the forces in a horizontal direction, and equate to zero.

(*c*) Find the sum of the moments of the forces about any point in the plane, and equate to zero.

(*d*) Solve the equations obtained in this way.

Full use should be made of sketches to aid in the elucidation of problems.

If a body is constrained in such a way that it can have motion of *rotation* but not motion of *translation*, then it is in equilibrium unless the forces acting upon it reduce to a couple.

EXAMPLES IV

(*For Hints on Working Examples, see page* 21.)

1. *Three forces, P, Q, and R, acting at one point are in equilibrium. P is a force of* 35 *poundals, acting vertically downwards, Q is a force of* 40 *poundals, and R is a force of* 32 *poundals. Find the inclination of Q and R from the vertical.*

First we draw the triangle of forces. To do this we set off along a vertical line a distance AB to represent, to a suitable scale, the vertical force P, of which we know both the magnitude and the direction.

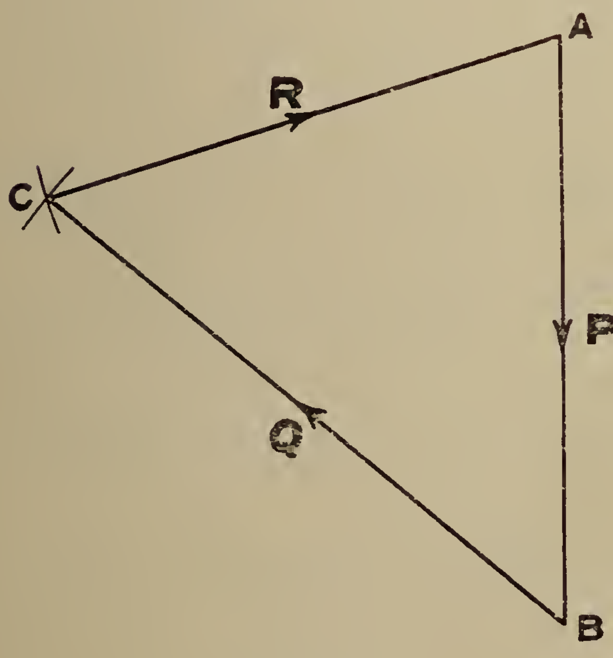


FIG. 30.

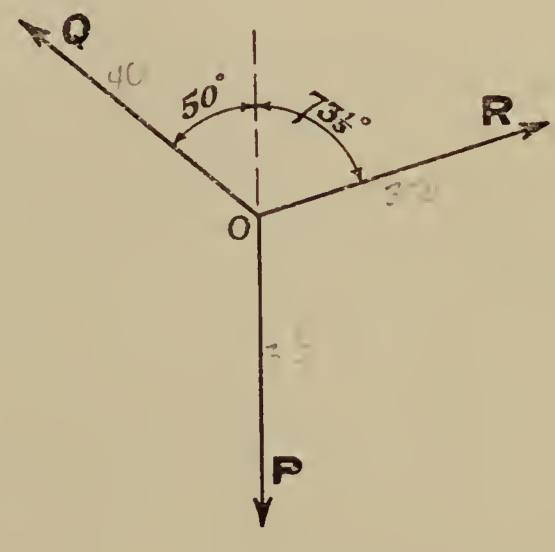


FIG. 31.

Then, we draw arcs from A and B to intersect in C, the radius of one arc representing to scale the magnitude of the force Q, and the radius of the other arc the magnitude of the force R. Then ABC (Fig. 30) is the triangle of forces for P, Q, and R, BC representing the force Q, and CA the force R.

We can now draw from any point O (Fig. 31), lines parallel to AB, BC, and CA, to represent the three forces acting at a point. Measuring the angles, we find that the line of action of Q is at an angle of 50 degrees from the vertical, and that the line of action of R is at an angle of $73\frac{1}{3}$ degrees from the vertical.

2. *A lamp of 2·7 pounds-weight is suspended from the ceiling of a room by two cords, one of which, 3 feet in length, is fixed to a point A, and the other, 2 feet 3 inches in length, is fixed to a point B, 3 feet 6 inches from A. Find the tension in each cord.*

First let us set out, to scale, as accurately as possible, a diagram of the arrangement (Fig. 32).

It will be seen that, at the point C, we have three forces in equilibrium, viz., the weight

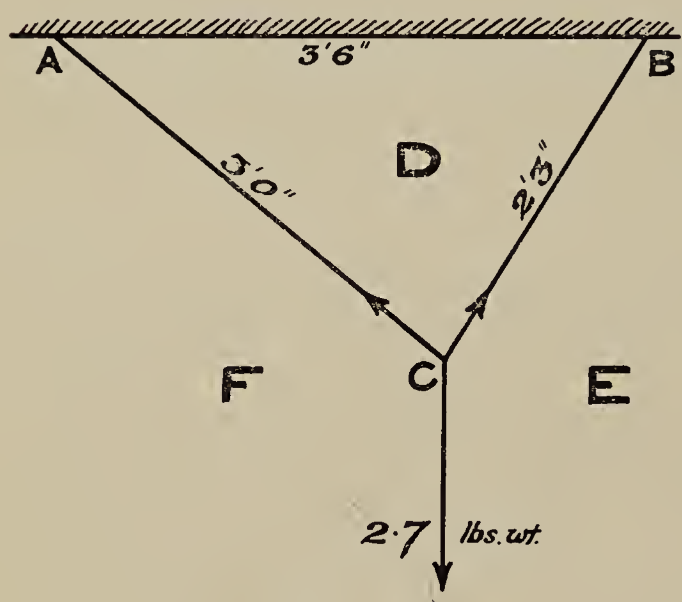


Fig. 32.

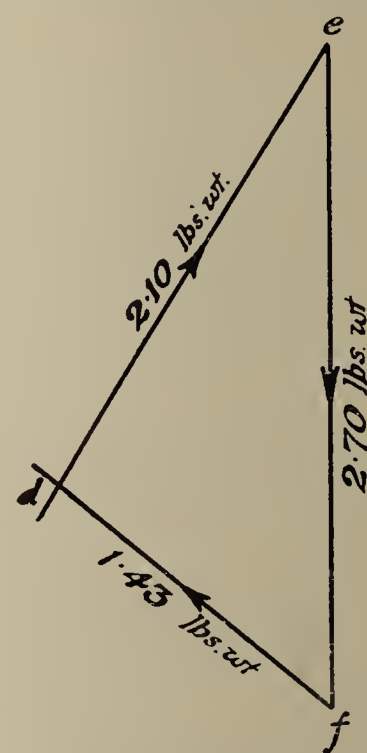


Fig. 33.

of the lamp acting downwards and the tensions in the cords acting upwards. We can therefore draw the triangle of forces for the point C.

Lettering the *spaces* in the diagram, D, E, and F, as shown, we draw ***ef*** (Fig. 33), to represent to scale the weight of the lamp (using as large a scale as we can).

Next we draw a line through ***e*** parallel to the force in the cord DE, i.e., parallel to the cord between the spaces D and E.

Finally we draw a line through ***f*** parallel to FD to cut the line through ***e*** in ***d***.

Then ***def*** is the triangle of forces for the point C.

Measuring ***de*** we find that the force in the cord DE is 2·10 pounds-weight.

Similarly ***fd*** represents a force of 1·43 pounds-weight in the cord FD.

Therefore we have :—

Tension in cord from A = 1·43 pounds-weight.
Tension in cord from B = 2·10 pounds-weight.

Note.—The method of lettering the *spaces* between the forces, employed above, is known as **Bow's Notation.** It is extremely useful and should be carefully studied.

3. *Three strings, OA, OB, and OC, meeting at a point O, are in equilibrium. If the angle AOB* = 100 *degrees, the angle BOC* = 150 *degrees, and the tension in the string OB is* 15 *pounds-weight, find the tensions in the other two strings.*

First we make a diagram (Fig. 34), showing the three strings at the correct angles (angle COA = 360° — 150° — 100° = 110°) and letter the spaces between the strings. ("Bow's Notation.")

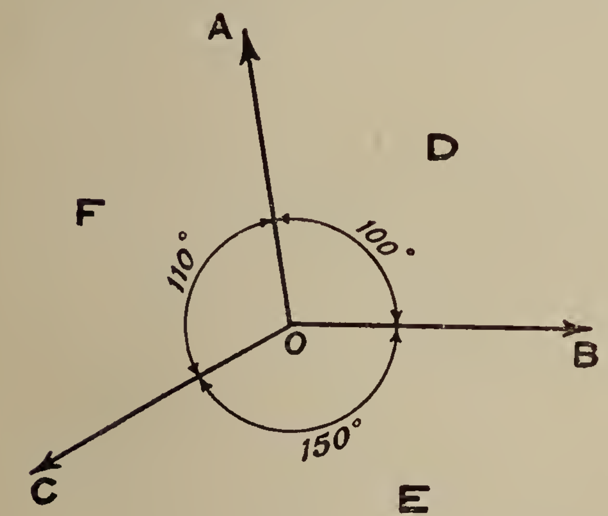
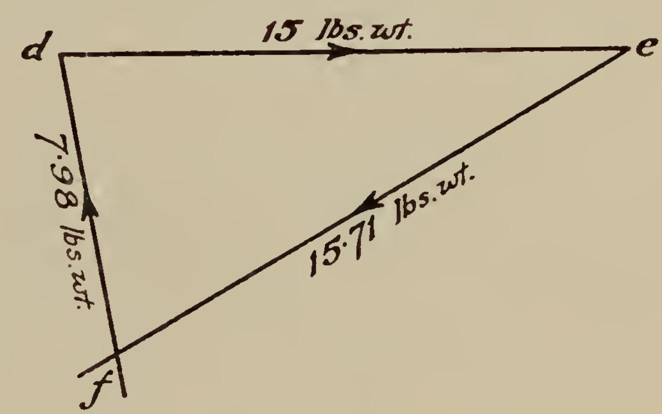


FIG. 34.



TRIANGLE OF FORCES.

FIG. 35.

Next we draw a line *de* (Fig. 35), to represent to some suitable scale the tension in the string DE (i.e., the string OB). Then through *d* we draw a line *df* parallel to DF (i.e., to OA), and through *e* a line *ef* parallel to EF (i.e., to OC). These lines *df* and *ef* meet in *f*, and represent in magnitude and direction the tensions in the corresponding strings.

Measuring the lines, we find that :—

Tension in the string OA = 7·98 pounds-weight.
Tension in the string OC = 15·71 pounds-weight.

4. *A weightless rod AB,* 26 *inches in length, is pulled by a force of* 35 *poundals at A, perpendicular to the length of the rod, and a force at a point C which is* 15 *inches from A. The line of action of this force is at an angle of* 60 *degrees from AC, and is on the opposite side from the first force. Find the magnitude of this force, and the magnitude and direction of the force which must be applied at B in order to keep the rod in equilibrium.*

Since the three forces are in equilibrium, and are not all parallel, therefore they must all pass through the same point. This point must be the point D where the forces acting at A and C intersect (Fig. 36).

Let β be the angle ABD.

Then AD $= 26'' \tan\beta = 15'' \tan 60°$,

whence $$\tan\beta = \frac{15 \times \sqrt{3}}{26}$$
$$= 1{\cdot}00$$
$$= \tan 45^\circ$$
which gives the direction of the force at B.

Having now the *directions* of all the forces, and the *magnitude* of one, we can obtain the magnitudes of the other two, either by the Triangle of Forces or by the Principle of Moments. Let us employ each method in turn.

Taking moments about B, we have :—

Force at C $\times$ 11″ sin 60° = 35 poundals $\times$ 26″,

whence $$\text{Force at C} = \frac{35 \text{ poundals} \times 26''}{11'' \times {\cdot}866}$$
$$= 95{\cdot}5 \text{ poundals.}$$

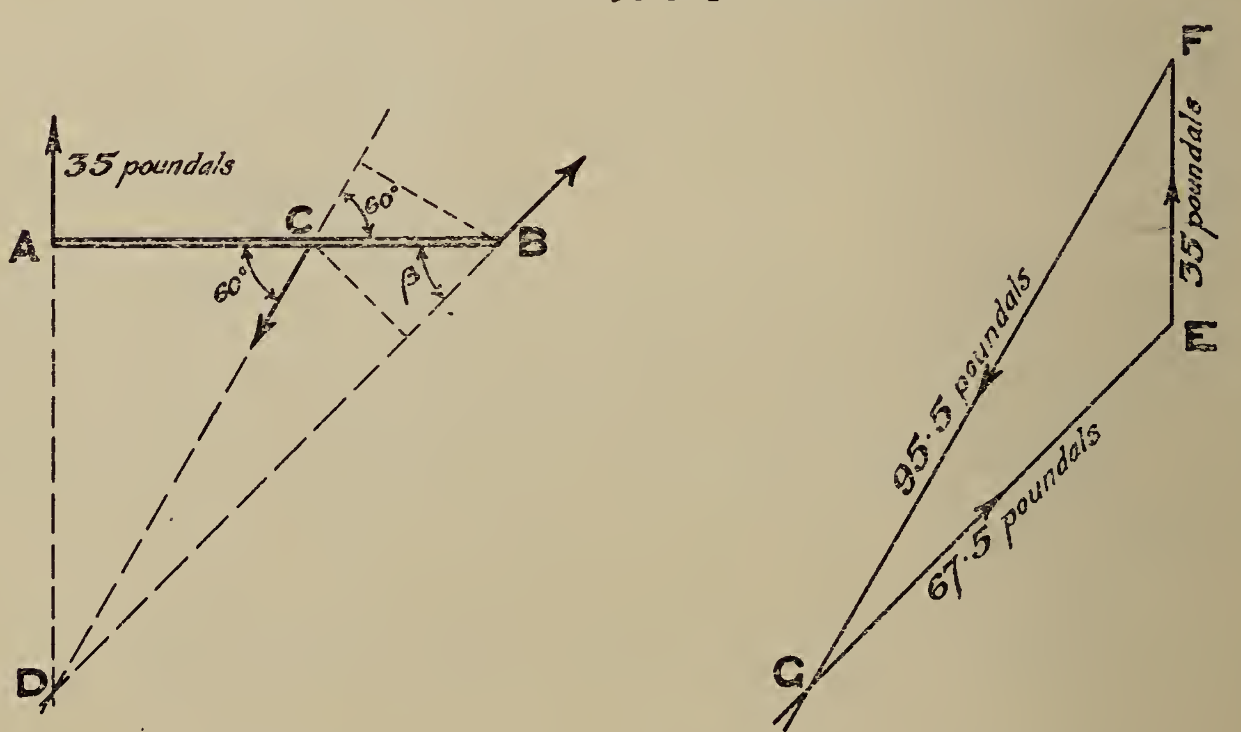


Fig. 36. Fig. 37.

Taking moments about C, we have :—

Force at B $\times$ 11″ sin 45° = 35 poundals $\times$ 15″,

whence $$\text{Force at B} = \frac{35 \text{ poundals} \times 15''}{11'' \times {\cdot}707}$$
$$= 67{\cdot}5 \text{ poundals.}$$

Drawing the Triangle of Forces (Fig. 37) we obtain the same result.

5. *A uniform bar AB of length* 3 *feet* 6 *inches and weight* 12 *pounds, is kept in a horizontal position by three strings, two of which are attached to A and one to B. One of the strings attached to A is vertical and has a*

tension of 8 pounds-weight. The string attached to B is inclined at an angle of 45 degrees (outwards) from AB. Find the direction and tension of the other string attached to A.

Let T_1 be the tension in the string at B, and T_2 the tension in the second string at A. Also let θ be the angle of inclination of the latter string to AB, as shown in Fig. 38.

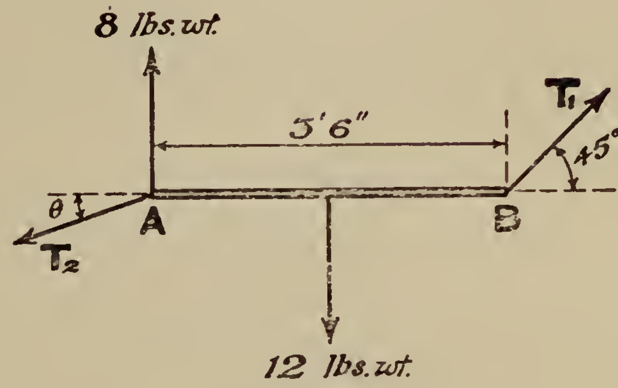


Fig. 38.

Taking moments about A

$$T_1 \times 3'6'' \times \cos 45^\circ = 12 \text{ pounds-weight} \times 1'9'',$$

whence
$$T_1 = \frac{12 \text{ pounds-weight} \times 1'9''}{3'6'' \times \cdot 707}$$
$$= 8\cdot 48 \text{ pounds-weight.}$$

Resolving horizontally

$$T_2 \cos \theta = T_1 \cos 45^\circ$$
$$= 8\cdot 48 \text{ pounds-weight} \times \cdot 707$$
$$= 6 \text{ pounds-weight.}$$

Resolving vertically

$$T_2 \sin \theta = T_1 \sin 45^\circ + 8 \text{ pounds-weight} - 12 \text{ pounds-weight}$$
$$= (6 + 8 - 12) \text{ pounds-weight}$$
$$= 2 \text{ pounds-weight.}$$

Therefore,
$$\tan \theta = \frac{T_2 \sin \theta}{T_2 \cos \theta}$$
$$= \frac{2 \text{ pounds-weight}}{6 \text{ pounds-weight}}$$
$$= \tfrac{1}{3}$$
$$= \tan 18^\circ 26',$$

so that
$$\theta = 18^\circ 26'$$

and
$$T_2 = \frac{2 \text{ pounds-weight}}{\sin \theta}$$
$$= \frac{2}{\cdot 3162} \text{ pounds-weight}$$
$$= 6\cdot 32 \text{ pounds-weight.}$$

i.e., the string attached at A is inclined at an angle of 18° 26′ to the bar, and the tension in it is 6·32 pounds-weight.

6. *A uniform rod AB,* 4 *feet in length and of* 8 *pounds-weight, is held at an angle of* 30 *degrees from the horizontal by means of* 3 *strings. One string is attached to the lower end A, at an angle of* 100 *degrees from the rod. Another string is attached to B and is in line with the rod. The third string is vertical and is attached to the rod at a point C, one foot from B. Find the tensions in the strings.*

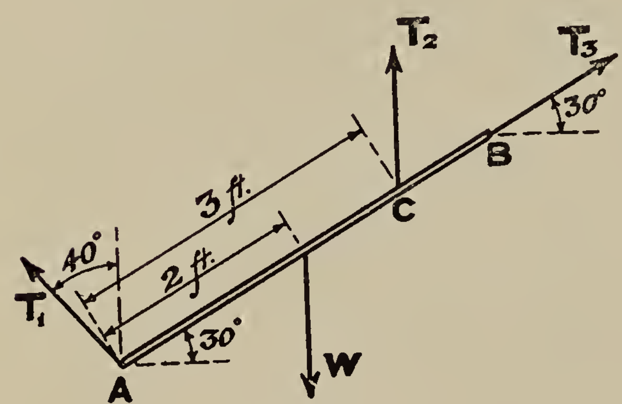


FIG. 39.

The first thing to do is to make a sketch (Fig. 39) and mark on it the information given in the question.

Let W = weight of rod in pounds-weight.
T_1 = tension in string at A.
T_2 = tension in string at C.
T_3 = tension in string at B.

Resolving vertically

$T_1 \cos 40° + T_2 + T_3 \sin 30° = W$

that is $\cdot 766\ T_1 + T_2 + \cdot 500\ T_3 = 8$ pounds-weight . . . . (1)

Resolving horizontally

$$T_1 \sin 40° = T_3 \cos 30°,$$

that is $\cdot 643\ T_1 = \cdot 866\ T_3$ . . . . (2)

Taking moments about A

$$T_2 \times (3 \text{ feet} \cos 30°) = W \times (2 \text{ feet} \cos 30°),$$

whence $3.T_2 = 2.W$

and $T_2 = \frac{2}{3}.8$ pounds-weight

$= 5\cdot 33$ pounds-weight . . . (3)

From (1) and (3)

$\cdot 766\ T_1 + \cdot 500\ T_3 = 8$ pounds-weight $- 5\cdot 33$ pounds-weight

$= 2\cdot 67$ pounds-weight . . . (4)

From (2) $\cdot 643\ T_1 - \cdot 866\ T_3 = 0,$

whence, multiplying by $\frac{\cdot 500}{\cdot 866}$ throughout,

$\cdot 371\ T_1 - \cdot 500\ T_3 = 0$ . . . . (5)

also $\cdot 766\ T_1 + \cdot 500\ T_3 = 2\cdot 67$ pounds-weight . . . (4)

$\therefore$ adding $1\cdot 137\ T_1 = 2\cdot 67$ pounds-weight,

whence $T_1 = 2\cdot 35$ pounds-weight . . . (6)

Substituting in (2)

$\cdot 643 \times 2\cdot 35$ pounds-weight $= \cdot 866\ T_3$,

whence $$T_3 = \frac{\cdot 643}{\cdot 866} \times 2\cdot 35 \text{ pounds-weight}$$
$$= 1\cdot 74 \text{ pounds-weight} \quad . \quad . \quad . \quad (7)$$

Therefore we have :—

Tension in string at A = 2·35 pounds-weight.
Tension in string at B = 1·74 pounds-weight.
Tension in string at C = 5·33 pounds-weight.

NOTE.—Further worked examples involving questions of equilibrium will be found after later chapters.

7. Three forces, P, Q, and R, acting at one point, are in equilibrium. P is a force of 12 pounds-weight acting horizontally, Q is a force of 17½ pounds-weight, and R is a force of 8 pounds-weight. Find the inclination of Q and R from the horizontal.

8. A load of 45 pounds-weight is suspended from a horizontal beam by two light chains, one of which, 5 feet in length, is fixed to a point A on the underside of the beam, and the other, 3 feet 9 inches in length, is fixed to a point B, 7 feet 3 inches from A. Find the tension in each chain.

9. Three light cords, OA, OB, and OC, meeting at a point O, are in equilibrium. If the angle AOB is 85 degrees, the angle BOC is 140 degrees, and the tension in the cord OB is 87 poundals, find the tensions in the other two cords.

10. What force will be required to balance a vertical force of 2½ tons-weight and a horizontal force of 4¼ tons-weight, both acting at the same point ?

11. A horizontal rod, AB, 32 inches in length and weighing 12 pounds, is pulled by a vertical upward force of 7 pounds-weight at A, and a force at a point C which is 12 inches from A. This force acts downwards and outwards at an angle of 45 degrees from AC. Find the magnitude of this force and the magnitude and direction of the force which must be applied at B, in order to keep the rod in equilibrium.

12. Forces of 156 pounds-weight, 168 pounds-weight, and 234 pounds-weight, acting at one point are in equilibrium. Find the angles between their lines of action.

13. Determine the magnitudes of the vertical and horizontal forces required to balance a force of 452 pounds-weight acting in a direction making an angle of 83 degrees with the vertical.

14. A beam, AB, 20 feet in length is supported at each end, and

carries the following loads: $2\frac{1}{2}$ tons-weight evenly distributed (including the weight of the beam); $1\frac{3}{4}$ tons-weight acting vertically at a point 4 feet from A; $3\frac{1}{4}$ tons-weight acting vertically at the centre of the beam; and 4 tons-weight at a point C, 5 feet from B, and acting at an angle of 60 degrees from BC. If the reaction at B is vertical, find its magnitude and the magnitude and direction of the reaction at A.

15. A bar AB, five feet in length and weighing $\frac{1}{2}$ cwt., is kept in a horizontal position by three wires in the vertical plane through AB. One of the wires is attached to the end A; it is horizontal and has a tension of 15 pounds-weight. The second wire is attached to B and is vertical. The remaining wire is attached to the bar at a point C, one foot from A. Find the tension in the wire attached at B, and the direction and tension of the wire at C.

16. A uniform pole AB, 25 feet in length and weighing 60 pounds, is hinged at A to a vertical wall, and supported by a light chain, 33 feet in length, which is fastened to B and to a point C, 15 feet above A. A load of 29 pounds-weight is suspended from B. Find the tension in the chain, and the magnitude and direction of the reaction at the hinge.

17. Two smooth cylinders, A and B, each weighing 84 pounds, rest on the ground side by side and in contact. If a third similar cylinder, C, be placed on top and parallel with them, what is the least horizontal force applied to A and B which will prevent motion?

18. P is a horizontal force. Q is a force which cuts the horizontal plane containing the force P at a point O which is not on the line of action of P. Prove that P and Q cannot be combined into a single force.

19. Find the equilibrant for a force of 224 pounds-weight in a direction due North, and a force of 317 pounds-weight in a South-Easterly direction.

20. A pole AB, 18 feet 6 inches in length and weighing 44 pounds, is used for lifting goods on a wharf. The end A is hinged to the ground and the end B is attached to a rope 24 feet in length, the other end of which is fastened to the ground at a point 10 feet behind A. Find the force in the rope when a load of one cwt. is hung from B, and the reaction at the hinge.

21. A uniform beam AB, 15 feet in length, slopes upwards from A at an angle of 30 degrees from the horizontal, and carries vertical loads, of 16 cwt. each, at A and at a point C which is 4 feet from B. The beam is supported at B and at a point D which is 3 feet from A. If the reaction at D is normal to the beam, find its magnitude, and the magnitude and direction of the reaction at B. The beam weighs 28 pounds per foot of length.

CHAPTER 5: CENTRE OF GRAVITY

EVERY rigid body may be considered to be composed of a very large number of particles of matter, the positions of which with regard to each other are definitely fixed.

The earth attracts each of these particles with a force of which the *magnitude* is proportional to the mass of the particle, the *direction* is vertical, and the *point of application* is at the centre of the particle. All these little forces together constitute the *weight* of the body.

We may say, then, that the **weight** of a body is the resultant of a system of parallel forces acting on the particles composing the body. Clearly this resultant will be vertical in direction, and, like any other force, will have some definite line of action. We shall see presently that it also has a definite point of application.

Let us make sure that we understand exactly what is meant by the term *point of application*. Suppose that when the body is in some particular position we mark on it the line of action of its weight, say AB, which will of course be a vertical line. Now suppose that we turn the body in such a way that the line AB marked on the body is no longer vertical. Gravity will continue to act upon the body with a force whose magnitude and direction are unchanged, but the line AB, being not now vertical, will no longer be the line of action of the weight of the body.

It will be found that the new line of action, say CD, of the weight will intersect the line AB at some point which we will call G.

If we turn the body again, so that neither the line AB nor the line CD is vertical, then the line of action of the weight will be along some third line, say EF, and it will be found that this line EF will also pass through the point G (Fig. 40). If we continue to move the

body into any number of different positions, in any way we please, we shall find that in every case the line of action of the weight of the body will pass through this same point G.

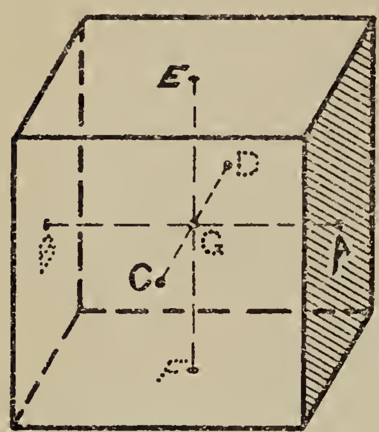


Fig. 40.

This definite point, through which in every position of the body the line of action of its weight passes, is the *point of application* of the force of gravity upon the body, and is known as the **centre of gravity** of the body.

CENTRE OF MASS. Now suppose that we could concentrate the whole mass of the body at some one point, in such a way that the point of application of the force of gravity upon it would be unchanged, then it is evident that this point of concentration would have to be the point which we have just named the centre of gravity. Because it is the point at which we may consider the whole mass of the body to be concentrated, this point is also known as the **centre of mass** of the body.

The centre of mass of a body and its centre of gravity are therefore at the same point, but it should be realised that the two terms do not therefore mean the same. One is the point at which the *mass* of the body may be supposed to be concentrated: the other is the point of application of the *weight* of the body.

PROOF OF CENTRE OF GRAVITY. We have seen what is meant by the term *centre of gravity,* but we have had no proof that such a point really exists. We must satisfy ourselves, therefore, that every rigid body does possess a centre of gravity; that is, that the point of application of the force of gravity upon a body is the same for all positions of the body.

Suppose that m_1, m_2, m_3, m_4 . . . . . (Fig. 41) are particles of a rigid body, and that w_1, w_2, w_3, w_4 . . . . . are the weights of the particles, acting in each case vertically downwards.

The resultant of w_1 and w_2 will be a force R_1 equal

in magnitude to $(w_1 + w_2)$, and also acting vertically downwards. The line of action of R_1 will cut the line joining m_1 and m_2 at a definite point G_1, such that the distance m_1G_1 is to the distance m_2G_1 as w_2 is to w_1. Now if the body be turned so that the force of gravity acts on the particles m_1 and m_2 as shown by w_1' and w_2', then the resultant of the two forces will act as shown by R_1', and its line of action will still cut m_1m_2 at the same point G_1. G_1 is therefore the point of application of the resultant R_1 of the two forces w_1 and w_2 however the body may be turned.

Having thus compounded the forces w_1 and w_2 into a single force R_1, having its point of application at G_1, we may then repeat the process and compound together the forces R_1 and w_3. We obtain as their resultant a force R_2 parallel to all the other forces, and having its point of application on the line joining G_1 and m_3 at some definite point G_2.

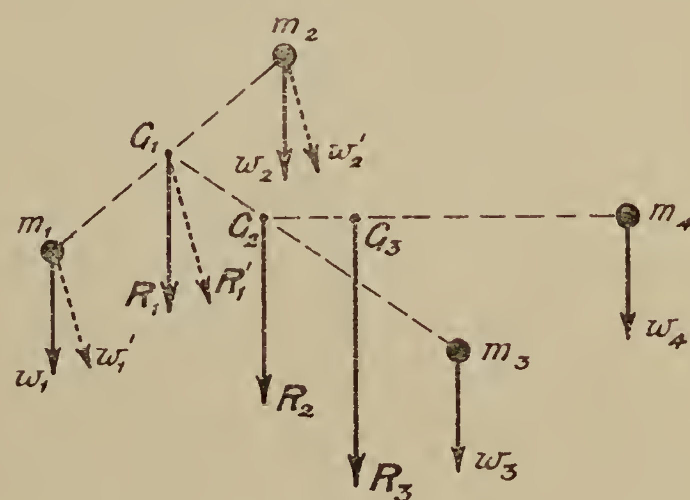


FIG. 41.

This force R_2 may then, in turn, be compounded with the force w_4 to obtain a resultant R_3 having its point of application at G_3 on the line G_2m_4.

Evidently, if we repeat the process with every one of the small forces acting upon the particles of the body, until we have compounded them all together, we shall obtain a final resultant R, of which the *magnitude* will be equal to the sum of the forces, the *direction* will be the same as that of all the forces, since they are parallel, and the *point of application* will be some definite point G. This will be the **centre of gravity** of the body, and its position is therefore definitely fixed with regard to the body, irrespective of the position in which the body may be placed.

The *order* in which the forces are compounded together makes no difference to the final result. In whatever order we take them, we shall obtain the same resultant acting at the same point.

CENTRE OF PARALLEL FORCES. The weights of the particles composing a body constitute a **system of parallel forces,** each of them being vertical. We can extend the reasoning we have employed above to *any* system of parallel forces. If, however, the forces with which we are dealing are not due to gravity, then the point of application of their resultant cannot be termed the centre of gravity: we call it simply the **centre of a system of parallel forces.**

We can say, then, that the centre of a system of parallel forces is the point of application of their resultant. In the particular case where the parallel forces are the weights of the particles of a rigid body, the resultant is the whole weight of the body, and its point of application is the centre of gravity of the body.

CENTRE OF AREA. The expression *centre of gravity of an area* is sometimes used. Now an area has length and breadth but no thickness: it has therefore no volume or mass, and so cannot be acted upon by gravity. It is incorrect, then, to speak of its centre of gravity: we should rather say **centre of area** or **centroid.**

We can easily see why the wrong term is employed, if we consider the case of a **lamina,** such as a figure cut out of thin card. A lamina *is* a solid body, and therefore *does* possess mass, and *has* a centre of gravity. If, now, we could make this lamina thinner and thinner, it would ultimately cease to have any thickness at all, and would then be not a solid body but merely an area. The point in the area corresponding to the centre of gravity of the lamina is frequently called the centre of gravity of the area, whereas it should be called the centroid or centre of area. We shall understand therefore that when the centre of gravity of an area is men-

tioned, what is really meant is the centre of gravity of a lamina which is so thin that we can regard its thickness as negligible.

EXPERIMENTAL DETERMINATION OF CENTROID. We can readily find by experiment the centre of gravity of a lamina, and hence the centroid of the corresponding area.

Suppose that we have a figure of any shape cut out of thin card or tinplate, and wish to find its centre of gravity. If we stick a pin through it anywhere and stick the pin into a wall, so that the pin is horizontal and the card figure can turn freely about it, the card will swing to and fro and eventually come to rest in some definite position, and even if disturbed, will return to that position.

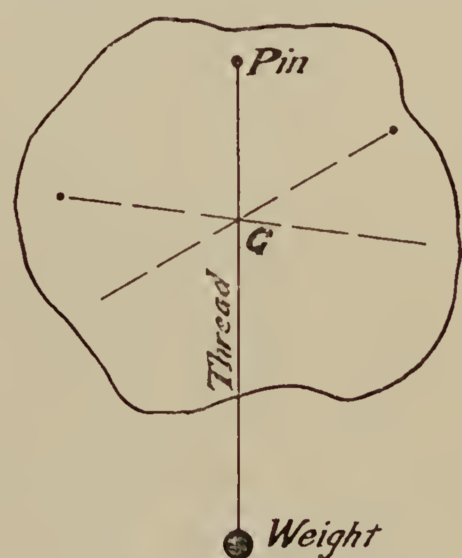


Fig. 42.

Now if we tie a small weight to a piece of fine thread and suspend it from the pin close up against the card, the thread will hang vertically, and we can mark on the card a vertical line through its point of suspension. Clearly, the only forces acting upon the card are its *weight*, acting vertically downwards through its centre of gravity, and the *reaction* of the pin on the card. When the card is at rest these forces are in equilibrium, and as there are only the two of them, they must be equal, opposite, and in the same straight line. The centre of gravity of the card must therefore lie somewhere on the line we have just marked upon it.

If we now take out the pin and reinsert it at some other point of the card, we can repeat the process and find another line somewhere on which also the centre of gravity of the card must lie. Evidently the actual position of the centre of gravity will be indicated by the intersection of the two lines we have drawn on the

surface, and will be at a point half-way through the thickness of the card at the point of intersection of the lines. The centroid of the figure represented by the surface of the card will be given by the actual intersection of the lines.

We can check the accuracy of our work by finding a third line by the same method, and if our experiment has been correctly performed, this third line will pass through the point of intersection of the other two lines, as shown in Fig. 42.

It should be noted that *the centre of gravity of a body does not necessarily lie in the body itself,* although its position with regard to the body is fixed. Similarly, the centre of area or centroid of a figure does not necessarily lie in the area of the figure, although its position with regard thereto is fixed. If the figure employed in the experiment just described was a circular ring, for example, we should find that the lines drawn would not meet anywhere on the card, but would intersect somewhere in the space enclosed by the inner circle; at the centre of the circles, in fact.

GRAPHICAL DETERMINATION OF CENTROID. We can find the centre of gravity of a lamina, or the centroid of an area, by the use of graphical methods instead of experiment, and these methods are the most generally useful. Let us consider a few of them.

The centroid of a straight line is obviously at its middle point: this hardly requires proof. Similarly, the centre of gravity of a very thin rod is at its middle point.

Now if we divide an area into strips so narrow that each may be regarded as simply a straight line, then the centroid of each strip will be at its middle point, and if these centroids all lie on one straight line, the centroid of the whole area will be somewhere on the same straight line. Similarly, if we divide a lamina into strips so narrow that each may be regarded as simply an extremely thin rod, then the centre of gravity of each rod will be

at its middle point, and if these centres of gravity all lie on the same straight line, then the centre of gravity of the whole lamina will be somewhere on the same line also.

Take, for example, a parallelogram, ABCD, as shown in Fig. 43. This may be supposed to be divided into a very large number of very narrow strips, parallel with two opposite sides. The centroid of each of these strips will be at its middle point, and the centroid of the whole area will be somewhere on the line joining the centroids of the strips, that is, somewhere on the line EF in the figure.

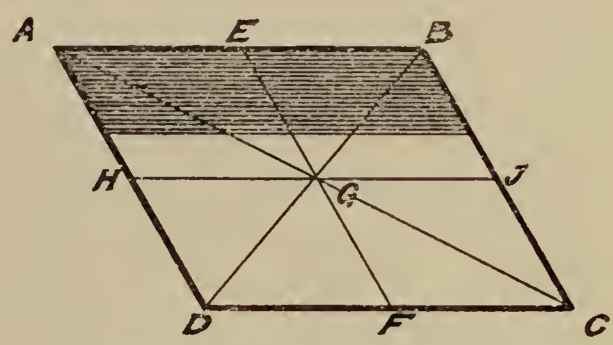


FIG. 43.

Now if we imagine the area divided into a similar series of strips parallel to the other sides, AD and BC, then we shall see that the centroid of the whole area must also lie on the line HJ, where H is the middle point of AD, and J is the middle point of BC. EF and HJ intersect at some point G, and this point, being the only one lying on both EF and HJ, must be the centroid of the whole area. It may easily be shown, by geometry, that the point G which we have obtained in this way, lies also on the intersection of the diagonals AC and BD of the parallelogram.

This method of finding the position of the centroid can be applied to other regular figures also. As another example we may take a triangle.

If we divide any triangle into a very large number of strips, each indefinitely narrow, parallel to the base, then the centroid must lie somewhere on the line joining the centres of the strips, that is, on the line joining the centre of the base to the opposite vertex. (AD in Fig. 44.)

Similarly, the centroid must lie on the line joining the middle point of *any* side to the opposite vertex,

and its actual position will be given by the point of intersection of the three lines, AD, BE, and CF, which can be drawn in this way. The perpendicular distance from any side of the triangle to the centroid, G, is therefore, by geometry, one-third of the perpendicular distance from that side to the opposite vertex.

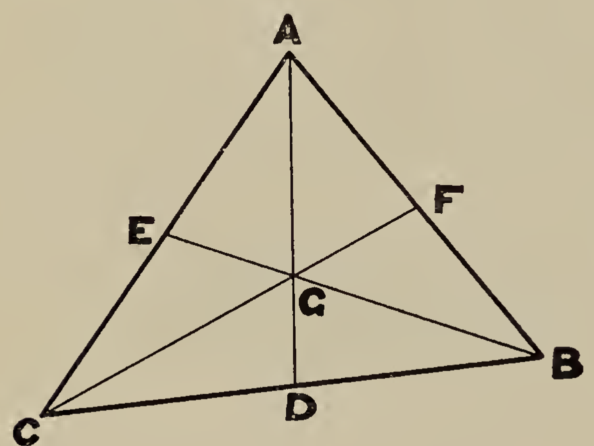


Fig. 44.

USEFUL RULES. In dealing with figures which are irregular, or more complicated than the simple cases considered above, the following points will be found useful and should be remembered:—

(i) The centroid of any symmetrical figure must lie somewhere on its axis of symmetry.

(ii) If a figure has two axes of symmetry, then its centroid must lie at their point of intersection.

(iii) If a figure can be divided into two parts and the centroid of each part found, then the centroid of the whole figure must lie somewhere on the line joining the centroids of the parts, and between them. Its position can be found by taking moments, for the moments of the two parts about the centroid of the whole must be equal and opposite.

(iv) If a figure can be considered to be the difference between two other figures, and the centroids of these other figures can be found, then the centroid of the original figure must lie somewhere on the line joining the centroids of the other figures *produced beyond the centroid of the larger figure.* Its position can be found, as before, by taking moments.

Let us take one example of the determination of the position of a centroid by the use of (iii) above.

ABCD (Fig. 45) is an irregular quadrilateral of which we wish to find the centroid. We first divide the figure

into two triangles ABC and ACD by joining A and C. The centroid of each of these triangles is readily found by the method previously explained, that is, by joining two of the vertices of each triangle to the middle points of the corresponding opposite sides. The intersections of these lines give the positions G_1 and G_2 of the centroids of the triangles. Join G_1 and G_2.

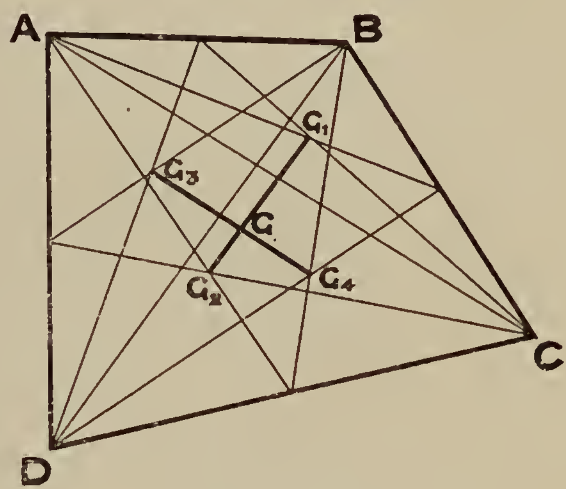


Fig. 45.

Now divide the figure again into two triangles, this time by a line drawn from B to D. Find the centroid G_3 of the triangle ABD, and the centroid G_4 of the triangle BCD. Then the centroid of the whole figure must lie on the line joining G_3 and G_4. Therefore its position is given by the intersection of the lines G_1G_2 and G_3G_4, that is, at the point G, as shown.

DETERMINATION OF CENTRE OF GRAVITY. In finding the position of the centre of gravity of a solid body we can employ somewhat similar methods. The following considerations are of service in many cases :—

(i) The centre of gravity of any symmetrical body of uniform density must lie somewhere in its plane of symmetry.

(ii) The centre of gravity of any body of uniform density which has two planes of symmetry must lie somewhere on the line formed by the intersection of those planes.

(iii) The centre of gravity of any body of uniform density which has more than two planes of symmetry must lie at the point where all the planes of symmetry meet.

(iv) If a body can be divided into two parts, and the centre of gravity of each part can be found, then the centre of gravity of the whole body must lie somewhere

on the line joining the centres of gravity of the parts, and between them.

(v) If a body can be considered to be the difference between two other bodies, and the centres of gravity of these latter can be found, then the centre of gravity of the actual body must lie somewhere on the line joining the centres of gravity of the other bodies, produced beyond the centre of gravity of the larger body.

In either of the two latter cases the actual position of the centre of gravity will be found by the application of the Principle of Moments, as explained when dealing with parallel forces in Chapter 3, the forces in these cases being the weights of the parts of the body. It will be observed that in doing this we employ the **first moment of a force** or **turning moment.** If we treated the determination as one of the centre of *mass* of the body, then we should have to use the **first moment of a mass.** In the same way, in finding the position of the centroid of an area, we employed the **first moment of an area.**

Considerations of symmetry show us that the centre of gravity of any regular body, such as a sphere or a cube, must lie at its geometrical centre, and the centre of gravity of a regular prism must be at the centroid of its middle cross-section ; provided, of course, that in each case the body is of uniform density throughout.

Variations and amplifications of these methods will enable us to determine the position of the centre of gravity in a great number of cases. The essential thing is to get a real grip of the *principles* involved, and then to practise their application in as many different ways as possible.

EQUILIBRIUM UNDER THE ACTION OF GRAVITY. If a body is acted upon only by the force of gravity, that is its weight, and an equal and opposite vertical reaction, then if these two forces are *not* in the same straight line they will form a couple which will cause the body to rotate until equilibrium is attained.

If the weight and the reaction act in the same straight line, then, clearly, they form a system of forces in equilibrium, and if the body is not disturbed by any other force it will remain at rest.

These conditions, viz., that the reaction on the body is equal and opposite to the weight and acts in the same straight line, are sufficient to show that the body is in equilibrium, but there are nevertheless very important differences between the equilibrium of one body under these conditions and that of another. These differences depend upon the relative position of the *points of application* of the two forces: that is, upon the relative position of the centre of gravity of the body and the point of application of the reaction.

Corresponding to these differences, equilibrium of this kind may be divided into three classes, viz., **stable, unstable,** and **neutral,** according to the relative positions of the two points of application. We may determine to which of these three classes a body belongs, by considering its behaviour when slightly displaced from its position of rest.

A body is said to be in **stable equilibrium** when, if it is slightly displaced from its position of rest, it returns to that position as soon as the displacing force is removed. An example is to be found in a lamp suspended from the ceiling, as shown in Fig. 46. If it is pushed slightly to one side and then released, it quickly comes back to its old position.

Fig. 46.

A body is said to be in **unstable equilibrium** when, if it is slightly displaced from its position of rest, it moves still farther from that position after the displacing force is removed. An example is to be found in a stick balanced on one end. So long as it is strictly vertical it is in equilibrium, but if it is tilted ever so slightly out of this position, it will fall right over.

A body is said to be in **neutral equilibrium** when, if it is slightly displaced from its position of rest, it neither

returns to that position nor moves farther from it, after the displacing force is removed. An example is to be found in a billiard-ball resting upon a level table. If it is displaced a little to one side, it will remain in its new position, neither returning to its former position, nor moving farther from it.

A *cone* may be placed in three different positions on a table to illustrate these three different kinds of equilibrium, as shown in Fig. 47. Resting on its base it is in *stable* equilibrium: balanced on its apex it is in *unstable* equilibrium: and lying on its side it is in *neutral* equilibrium.

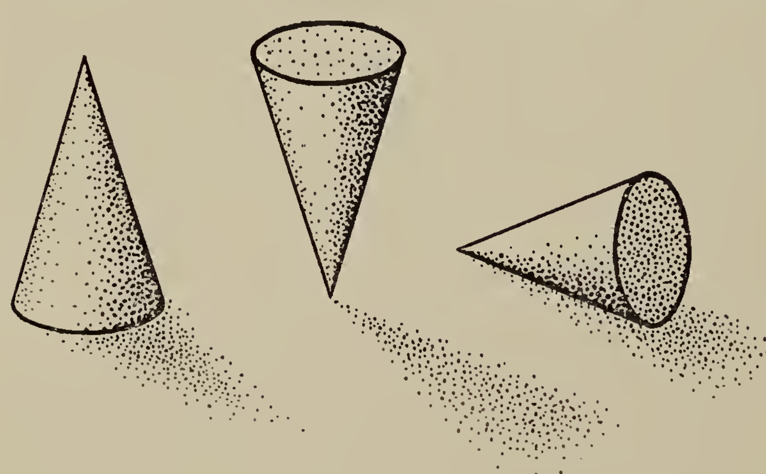


FIG. 47.

In general, a body is in *stable* equilibrium when its centre of gravity G is vertically *below* the point of application A of the balancing reaction, provided that point is fixed relative to the body and does not change with the slight displacement: it is in *unstable* equilibrium when its centre of gravity is vertically *above* the point of application of the reaction: and it is in *neutral* equilibrium when its centre of gravity *coincides* with the point of application of the reaction. (See Fig. 48.) It will be noticed, however, that this rule cannot be applied in all cases.

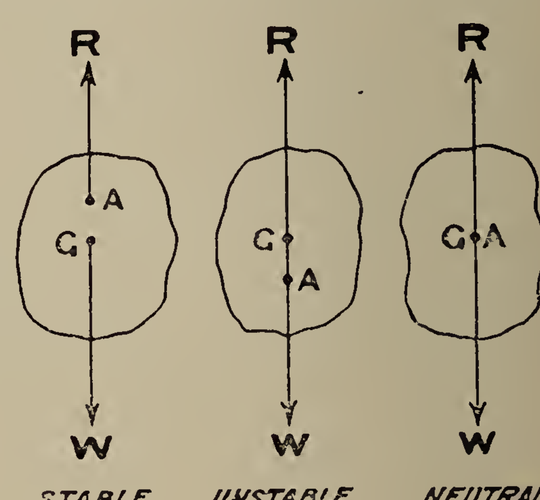


FIG. 48.

The fundamental distinction between the three cases is clearly, that in the case of stable equilibrium, the weight and the reaction form, as soon as the body is slightly displaced, a **righting couple** which tends to

restore the body to its original position, whereas in the case of unstable equilibrium, they form an **overturning couple** which tends still farther to displace the body, and in the case of neutral equilibrium, they do not form a couple at all but act always in the same straight line.

SUMMARY OF CHAPTER 5

The **centre of gravity** of a body is the point of application of its weight. The **centre of mass** of a body is the point at which we may consider the mass of the body to be concentrated. These two points coincide, because weight is always in proportion to mass.

Every rigid body has a centre of gravity of which the position with regard to the body is definitely fixed.

The **centre of a system of parallel forces** is the point of application of their resultant.

Corresponding to the centre of gravity of a solid body, a plane figure has a **centre of area** or **centroid.** This can be determined experimentally, or graphically, or by calculation.

If a plane figure has one or more axes of symmetry, then the centroid must lie on each of them, and hence at their point of intersection.

If a solid body of uniform density has one or more planes of symmetry, then its centre of gravity must lie in each of these planes, and hence on their line or point of intersection.

The centroid of a plane figure does not necessarily lie in the area of the figure, nor the centre of gravity of a solid body in the body, but may be at some point outside, whose position relative to the figure or body is definitely fixed.

Bodies which are acted upon only by their weight and an equal and opposite reaction in the same straight line, are in equilibrium, but the nature of this equilibrium depends upon the relative positions of the points of application of the two forces.

(*a*) If the centre of gravity of the body is below the point of application of the reaction, and the latter point is fixed relatively to the body, then the equilibrium is **stable.**

(*b*) If the centre of gravity of the body is above the point of application of the reaction, and the latter point is fixed relatively to the body, then the equilibrium is **unstable.**

(*c*) If the centre of gravity of the body coincides with the point of application of the reaction, then the equilibrium is **neutral.**

EXAMPLES V

(*For Hints on Working Examples, see page* 21.)

1. *A rod,* 5 *feet in length, and of uniform cross-section, is composed of equal lengths of wood and iron. If the specific gravity of the wood is* ·56 *and that of the iron is* 7·7, *find the position of the centre of gravity of the rod.*

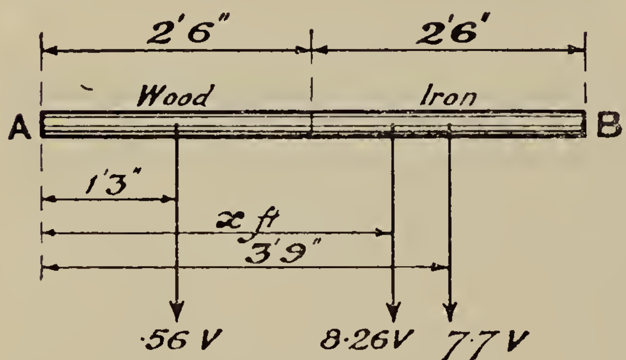


FIG. 49.

The volume of the wood will be the same as that of the iron: let the volume of either be V.

Then the weight of the wooden portion of the rod will be proportionate to ·56V, and that of the iron portion will be proportionate to 7·7V, so that the weight of the whole rod will be proportionate to (·56 + 7·7)V, that is to 8·26V.

Taking moments about the wooden end, A :—

$$8{\cdot}26V \times x = ({\cdot}56V \times 1'\ 3'') + (7{\cdot}7V \times 3'\ 9'')$$
$$= {\cdot}70V + 28{\cdot}9V$$
$$= 29{\cdot}6V,$$

whence
$$x = \frac{29{\cdot}6}{8{\cdot}26} \text{ feet}$$
$$= 3{\cdot}58 \text{ feet}$$
$$= 3 \text{ feet } 7 \text{ inches,}$$

i.e., the centre of gravity of the rod is at a distance of 3 feet 7 inches from the wooden end of the rod.

2. *A uniform rod AB,* 5 *feet in length and of* 10 *pounds mass, has a series of masses fixed to it as follows:* 2 *pounds at A,* 3 *pounds at* 1 *foot from A,* 4 *pounds at* 2 *feet from A,* 5 *pounds at* 3 *feet from A,* 2 *pounds at* 4 *feet from A, and* 8 *pounds at B. Find the point at which the rod will balance.*

First let us set out on a sketch all the information available.

Let R = resultant weight, acting at a distance x feet from A.

= 2 + 3 + 4 + 10 + 5 + 2 + 8 pounds-weight.

= 34 pounds-weight.

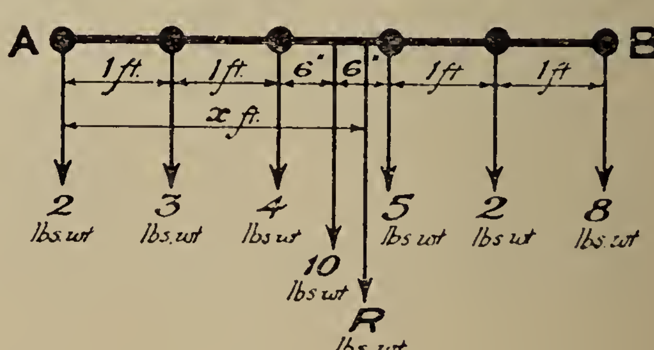


FIG. 50.

Taking moments about A :—

R pounds-wt. $\times$ x feet $= (2 \times 0) + (3 \times 1) + (4 \times 2) + (10 \times 2\frac{1}{2})$
$+ (5 \times 3) + (2 \times 4) + (8 \times 5)$ pound-feet.
$= 0 + 3 + 8 + 25 + 15 + 8 + 40$ pound-feet.
$= 99$ pound-feet,

whence $x = \dfrac{99 \text{ pound-feet}}{34 \text{ pounds-weight}}$
$= 2{\cdot}91$ feet.

The rod will therefore balance about a point in it 2·91 feet from A.

3. *Two circles in the same plane have their centres one inch apart. The diameters of the circles are* $3\frac{1}{2}$ *inches and* $1\frac{1}{4}$ *inches respectively. Find the position of the centroid of the area between the two circles.*

Let A be the centre of the larger circle and B the centre of the smaller circle. Then G, the centroid of the given area, must lie on the line AB produced beyond A, since this line is an axis of symmetry.

Taking moments about G, we have :—

Area of larger circle $\times$ GA $=$ area of smaller circle $\times$ GB

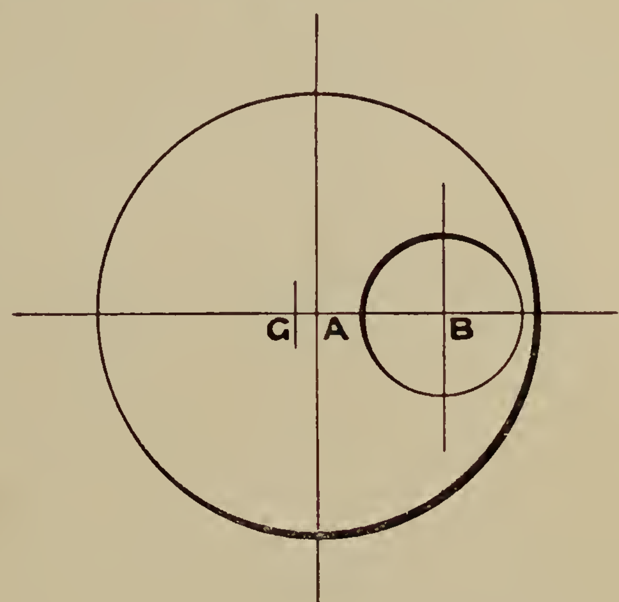


Fig. 51.

$$\frac{\pi}{4}(3\tfrac{1}{2})^2 \times GA = \frac{\pi}{4}(1\tfrac{1}{4})^2 \times GB$$
$$= \frac{\pi}{4}(1\tfrac{1}{4})^2 \times (GA + 1'').$$
$$\therefore \quad \frac{49}{4}.GA = \frac{25}{16}(GA + 1'').$$

Multiplying by 16 throughout, we have :—

$$196.GA = 25.GA + 25'',$$

whence $171.GA = 25''$

and $GA = \dfrac{25}{171}$ inch
$= {\cdot}146$ inch.

The centroid of the area between the two circles is therefore at a point ·146 inch from the centre of the larger circle on the opposite side from the centre of the smaller circle.

4. *ABCD is an irregular quadrilateral. $AB = 2$ inches, $BC = 2\frac{1}{2}$ inches, $CD = 3$ inches, $DA = 3\frac{1}{2}$ inches, and the angle $ABC = 120$ degrees. Find graphically the distance of the centroid from A and from C.*

First the figure must be drawn accurately to scale.

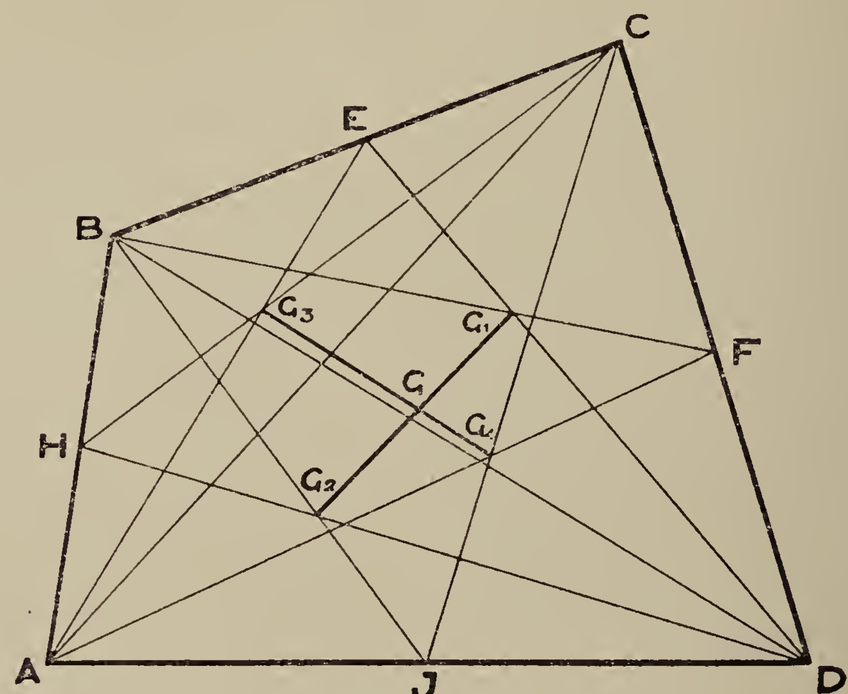


FIG. 52.

Next we divide it into two triangles, ABD and BCD, by joining B to D.

Bisecting BC in E and CD in F, and joining BF and DE, their intersection gives the centroid G_1 of the triangle BCD.

Similarly, bisecting AB in H, and AD in J, and joining DH and BJ, their intersection gives the centroid G_2 of the triangle ABD. Then the centroid of the whole figure must lie somewhere on the line G_1G_2.

We now divide the figure into two other triangles, ABC and CDA, by joining A to C, and obtain the centroids G_3 and G_4 of these triangles by the same method as above. Then the centroid of the whole figure must lie somewhere on the line G_3G_4.

Consequently the centroid of the whole figure is at the point of intersection, G, of the lines G_1G_2 and G_3G_4, and the required distances are $AG = 2{\cdot}07$ inches and $CG = 1{\cdot}95$ inches.

5. A bar of 1-inch square cross-section and 6 feet 6 inches in length is composed of equal lengths of lead, brass, and aluminium. Taking the weight of a cubic foot of each material as 712 pounds, 492 pounds, and 162 pounds, respectively, determine the position of the centre of gravity of the rod.

6. An iron rod weighing 12 pounds and of uniform cross-section carries a solid iron ball at each end. One of the balls weighs 2 pounds and the other weighs $5\frac{1}{2}$ pounds: the distance between the balls, centre to centre, is 18 feet 4 inches. Find the position of the centre of gravity.

7. A uniform rod AB, 4 feet in length and of $7\frac{1}{4}$ pounds mass, has the following masses attached to it: $1\frac{1}{2}$ pounds at A; 5 pounds at 1 foot 8 inches from A; 3 pounds at the centre; and 4 pounds at B. Find the position of the centre of gravity of the rod and attachments.

8. A square ABCD of 4-inch side contains a circle of $1\frac{1}{2}$-inch diameter with its centre midway between A and the centre of the square. Find the position of the centroid of the area which lies between the circle and the square.

9. An irregular quadrilateral ABCD has the following dimensions: AB = 3 inches; BC = $2\frac{1}{2}$ inches; CD = $3\frac{1}{2}$ inches; DA = 2 inches; and BD = 4 inches. Determine graphically the distance of the centroid of the quadrilateral from A and from B.

10. A rectangular card is $5\frac{1}{2}$ inches in length and $3\frac{1}{2}$ inches in breadth, and has a circular hole cut in it $1\frac{3}{4}$ inches in diameter. If the card is of uniform thickness, and the centre of the circle is midway between the two long edges, and $1\frac{1}{2}$ inches from one of the shorter edges, find where another circular hole of $2\frac{1}{4}$ inches diameter must be made, in order that the centre of gravity of the card may be equidistant from each corner.

11. A thin circular card 8 inches in diameter has an equilateral triangle of 3-inch side cut out of it, two of the sides of the triangle being radial to the circle. Find the position of the centre of gravity of the card.

12. Find the position of the centre of gravity of a solid cube of 4-inch edge, with another cube of double the density and 2-inch edge let into one corner.

13. ABCD is a square, of $3\frac{1}{2}$-inch side, on which is drawn a triangle ABE, such that E is 2 inches from D and $1\frac{3}{4}$ inches from C: find graphically the distance of the centroid of the triangle from A and from B.

14. A cube of $2\frac{1}{4}$-inch edge is fixed symmetrically to one end of a solid cylinder of the same material and of 2-inch diameter and 7 inches in length. Find the position of the centre of gravity of the resultant body.

15. A square of 6-inch edge and a circle of $1\frac{1}{2}$-inch diameter are cut out of uniform thin card and stuck together so that the circumference of the circle just touches one edge and one diagonal of the

square. Find the position of the centre of gravity, measured from the nearer sides of the square.

16. ABCD is an irregular quadrilateral; the angle DAB is a right-angle, and the lengths of the sides are as follows: AB = 3 feet 4 inches: BC = 5 feet: CD = 4 feet 4 inches: and DA = 2 feet 8 inches. Determine the perpendicular distance of the centroid of the quadrilateral from AB and from DA.

17. A uniform plank 16 feet in length and of 174 pounds mass carries a mass of 28 pounds at one end. If the plank is placed horizontally across a wall 18 inches in width, find the greatest and least distances that the unloaded end of the plank can project beyond the wall without overbalancing.

18. A solid cube of uniform density and of 6-inch edge is composed of eight equal, smaller, cubes. If one of these smaller cubes is removed, find the position of the centre of gravity of the remainder.

19. A triangle ABC has the side AB = 5 inches, the side BC = $3\frac{3}{4}$ inches, and the included angle = 60 degrees. Find the centre of gravity of masses of 3, 4, and 5 pounds placed at the points A, B, and C, respectively.

20. Determine the position of the centre of gravity of a box made from one-inch boarding, and of the following outside dimensions: length, one foot; breadth, 9 inches; height, 8 inches. The box is open at the top and has no lid.

21. A cube, 10 inches in edge, is built up from four equal square prisms of lead, copper, zinc, and aluminium, which are so arranged that the lead is in contact with the copper and the aluminium. The specific gravities are as follows: lead, 11·4; copper, 8·8; zinc, 7·0; aluminium, 2·6. Find in which metal the centre of gravity of the cube is situated and at what perpendicular distances from the nearer faces of the cube.

22. Find the centre of area of a figure ABCDEF of which all the angles are right-angles, and the dimensions are as follows: AB = 7 inches; BC = 2 inches; CD = $5\frac{3}{4}$ inches; DE = 3 inches; FA = 5 inches; and EF = $1\frac{1}{4}$ inches.

CHAPTER 6 : TRANSLATION

WE have already noticed (in Chapter 4) that there are two kinds of motion, either or both of which may be possessed by a body. The first is **translation** or motion in a straight line : the second is **rotation** or motion round a fixed axis. We have now to consider some properties of motion of translation : the corresponding properties of motion of rotation we shall defer until the next chapter.

Everybody knows what is meant by **speed.** If we want to define it, we may say that speed is *rate of motion*, that is, the speed of a body is the rate at which it is moving. Suppose, for example, that a motor-car is moving at such a rate that the total distance it covers in each hour is 20 miles, then we say that the speed of the car is 20 miles per hour. Now in referring to the speed of a car, or of any other moving body, we take no account of the *direction* in which the car has been moving : in fact, its direction may have changed repeatedly. So long as we know the distance covered in a given time, we do not concern ourselves with the direction of motion.

If, however, we want complete information about the motion of a body, then we must know not only its *speed* but also its *direction* of motion. We cannot say that we know all about the motion of a motor-car, for example, if we merely know that it is moving at 20 miles per hour : we must find out also in what direction it is going. If we are in a position to say, for instance, that it is going due North at a speed of 20 miles per hour, then our knowledge of its motion is complete.

Now these **two qualities of motion,** namely **speed** and **direction,** are not independent affairs having no connexion with each other, but are intimately and inseparably bound up with one another. We cannot imagine a body moving without having any direction

of motion ; nor can we think of motion without the idea of speed.

We therefore combine the two ideas of speed and direction in the one word **velocity.** If we want a formal definition, we can say that the velocity of a moving body is its *rate of change of position*, or *rate of displacement*, but we must never allow ourselves to forget that the term velocity includes direction as well as speed. We do not know the velocity of a body unless we know which way it is moving as well as how fast.

Clearly, *speed* is a **scalar** quantity, that is, it is a quantity which possesses *magnitude only* and not direction. *Velocity*, on the other hand, is a **vector** quantity, that is, it is a quantity which possesses *both magnitude and direction.*

COMPOSITION AND RESOLUTION OF VELOCITIES. Evidently, then, a velocity, being a vector quantity, can be completely represented by a straight line, the length of the line representing the magnitude of the velocity (i.e., its speed) and the direction of the line representing the direction of the velocity.

It follows, also, that if we wish to add two or more velocities together, we must do so in accordance with the laws of vector addition, which have already been explained in Chapter 2. It will be remembered that we added together or *compounded* forces by means of the Parallelogram of Forces. Similarly, we compound velocities by means of the **Parallelogram of Velocities** : the method is exactly the same in both cases. The reverse process of splitting a velocity up into components, or *resolving* it, is carried out by the use of the same theorem, in exactly the same way as we employed when resolving a force into components.

At first sight it may not be evident why we should want to add together velocities in different directions. An example will quickly make the matter clear.

Suppose that a steamer is travelling due East at a speed of, say, 10 miles per hour, along a river. Then,

of course, not only the steamer itself, but also its cargo, its crew, and its passengers, will all be carried along with a velocity of 10 miles per hour due East.

Now suppose that at some particular instant the boat is in the position represented by AB on the plan (Fig. 53), and that, at that instant, there is a passenger on the deck at the point marked C. Now if the passenger strolls across the deck at right angles to the length of the boat, that is, moving due North across the deck, then a few seconds later he will have arrived at the point D. But in the same interval of time that it has taken him to cross the deck, the steamer itself will have moved along the river to some fresh position A′B′. Therefore the actual position at which the passenger will have arrived will be not D but D′.

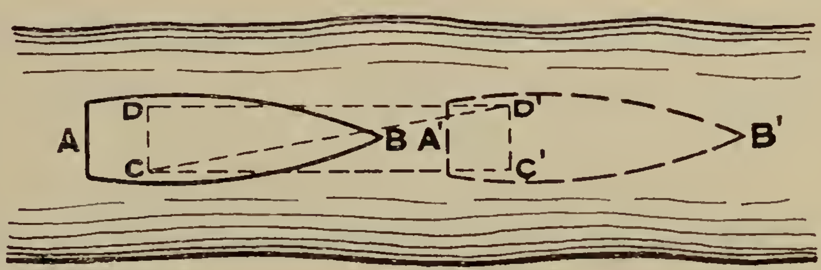


Fig. 53.

We see, then, that the motion of the passenger may be considered to be made up of two components. One is his own motion across the deck of the boat due Northwards at, say, 2 miles per hour, and the other is the motion which he shares with the boat, of, say, 10 miles per hour, due Eastwards. If he had only the former motion, that is to say, if he was walking across the deck but the steamer itself was not moving, then he would move from C to D only. If, on the other hand, the passenger stood still on one spot of the deck and the boat moved along the river, then he would move from C to C′ only.

Actually, both these movements take place, and the combined effect of them is that the passenger moves from C to D′, part of this change of position, or *displacement*, being due to his walking across the deck, and part to the steamer travelling along the river. If, then, we want to find the whole velocity of the passenger, we must compound together these two velocities, viz., 10

miles per hour due East, and 2 miles per hour due North. Doing this by means of the Parallelogram of Velocities, as shown in Fig. 54, we find that the *resultant velocity* of the passenger is about 10·2 miles per hour in a direction making an angle of about $11\frac{1}{4}$ degrees North of East.

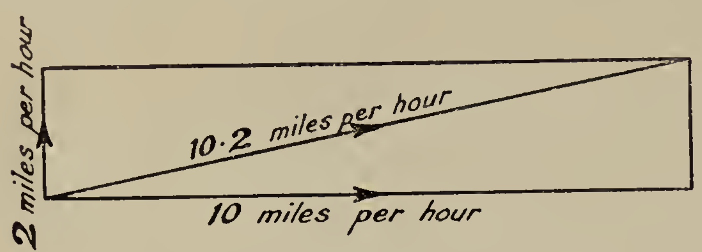


Fig. 54.

It may be noted that CDD′C′ in Fig. 53 is an instance of the **Parallelogram of Displacements.** Displacement is itself a vector quantity, and is therefore subject to the Parallelogram Law. In this particular case we have compounded together the two displacements of the passenger, the one across the river, represented by CD, and the other along the river, represented by CC′, the resultant displacement being represented in magnitude and direction by the diagonal CD′ of the parallelogram.

RELATIVE VELOCITY. It is sometimes convenient to consider the motion of a body compared with some other body only. For example, in the above case we may think only of the motion of the passenger across the deck, without regard to the motion of the vessel along the river. We term this the **relative velocity** of the passenger with regard to the steamer. Similarly, if the water of the river were flowing Westwards at a speed of 1 mile per hour, we could say that the velocity of the boat relative to the water was 11 miles per hour Eastwards. We could go on to calculate the relative velocity of the water with regard to the man, if there were any reason for so doing. *Relative velocity*, then, gives us the rate of displacement of one body with respect to another.

UNIFORM AND VARIABLE VELOCITY. The rate at which a body changes its position may be constant, or the rate itself may be changing: we say therefore that the velocity of a body is either **uniform** or **variable.**

If a body is moving with *uniform* velocity, then it passes through equal distances, all in the same direction, in equal intervals of time ; that is, its speed is constant, and its direction of motion is also constant. For instance, if a man walks steadily at 5 miles per hour along a straight road, we can say that he is moving with uniform velocity.

If a body is moving with *variable* velocity, then either the *speed*, or the *direction* of motion, or *both*, may be changing. If the body moves through unequal distances in equal intervals of time, then its speed is variable and therefore its velocity is variable also. If the speed is constant but the direction is changing, then again we have variable velocity. For instance, if a man starts walking at 5 miles per hour but slows down gradually, so that his pace falls to 4 miles per hour and gets slower and slower the farther he goes, then evidently his velocity is variable. Again, if his speed remains constant at 5 miles per hour but his direction varies, through the road he follows not being straight, then his velocity is variable.

We must never forget that *velocity only remains constant so long as both its magnitude and its direction remain unaltered*. A change of direction is a change of velocity just as truly as is a change of speed.

MEASUREMENT OF VELOCITY. A body has unit velocity when it passes through unit distance in unit time. We can therefore obtain a great variety of units of velocity by employing various units of distance and time : some of the units so obtained are very convenient for practical use, as, for example, a unit velocity of *one mile per hour* which is useful when dealing with moving vehicles ; but in Mechanics it is best, as far as possible, to employ systematic units. As already explained, systematic units are those derived in the simplest possible way from the three fundamental units, and consequently, for each system of units there is only one *systematic* unit of velocity.

In the F.P.S. (foot-pound-second) system, the systematic unit of distance or length is the foot, and the systematic unit of time is the second: therefore the systematic unit of velocity is a velocity of **one foot per second.**

In the C.G.S. (centimetre-gramme-second) system, the systematic unit of length or distance is the centimetre, and the systematic unit of time is the second: therefore the systematic unit of velocity is a velocity of **one centimetre per second.**

If a body moves with uniform velocity, then the magnitude of its velocity is the number of feet or the number of centimetres through which it moves in each second, according to the system that we are employing.

If, on the other hand, the velocity is varying, and the variation is due to changing speed, then the case is not quite so simple. We can deal with such a velocity in either of two different ways, the choice of a method depending upon the particular circumstances of the case with which we may be concerned. The alternatives are as follows:—

(*a*) We may determine the *actual* velocity of the body at any particular instant. Obviously this velocity will be different from that at the previous instant, and also from that at the subsequent instant.

(*b*) We may find the *mean* or average velocity of the body during some given period of time. We can do this by dividing the total distance travelled by the body by the total time occupied. For example, if a body has travelled 350 feet in 7 seconds, then its mean velocity is clearly equal to 350 feet divided by 7 seconds, i.e., to a velocity of 50 feet per second. This result is evidently quite independent of the *actual* velocity at any instant: indeed we have no information about the *actual* velocity during any part of the motion. It will be seen that the mean velocity is the same whether the body is moving with uniform or variable velocity, provided that the total distance and the time occupied are the same in each case.

DIMENSIONS OF VELOCITY AND SPEED. As we have just seen, we obtain the magnitude of a velocity by dividing the distance travelled by the time occupied. A distance is a length and is therefore of the dimensions **[L]**. A period of time is of the dimensions **[T]**. The dimensions of a velocity are therefore **[L]** divided by **[T]**, which we usually write in the form $[L][T]^{-1}$. The dimensions of speed are the same as those of velocity.

ACCELERATION. If a velocity is not uniform, then it must be changing either its magnitude, or its direction, or both. A change of magnitude may be either an increase or a decrease. In any case, we can state that in every interval of time there must be *some* change of velocity. The *rate* at which these changes take place is denoted by the term **acceleration.**

We may define acceleration therefore as the *rate of change of velocity* of a body, i.e., as the change of velocity it undergoes per unit time.

The velocity in a particular case may be changing uniformly, in which case the body has a *constant* acceleration: or it may be changing at a variable rate, in which case the body has a *variable* acceleration.

When acceleration is concerned with change of speed only, we may term it also either *positive* or *negative*. When the speed is increasing the body is said to have a positive acceleration: when it is decreasing the body is said to have a negative acceleration. A negative acceleration is sometimes called a *retardation*.

Since any change of velocity must be in some definite direction, the corresponding acceleration must be in the same direction. Clearly, therefore, *acceleration is a vector quantity*, so that it can be completely represented by a straight line, of which the direction gives the direction of the acceleration, and the length gives the magnitude of the acceleration. It follows also that we can compound together accelerations by means of the Parallelogram Law, and can split up an acceleration into components by the same means. The construction

for the **Parallelogram of Accelerations** is exactly the same as for the Parallelogram of Forces and the Parallelogram of Velocities already described.

We have seen that the idea of relative velocity is sometimes useful: so also we sometimes have occasion to speak of **relative acceleration.** Relative acceleration is the acceleration of one body considered with respect to another.

It will be noticed that there are distinct resemblances between the properties and methods of classification of *velocities* and those of *accelerations.* These resemblances depend upon the fact that each of these quantities is a *rate of change* and is a vector quantity. Velocity is rate of change of position: acceleration is rate of change of velocity. Clearly, acceleration is related to velocity in exactly the same way that velocity is related to position. Any properties therefore which are true for the relationship between velocity and position, are equally true for the relationship between acceleration and velocity. This is worth remembering.

ACCELERATION ALONG AND PERPENDICULAR TO PATH. When dealing with velocity we emphasized the fact that any change of direction of motion is as much a change of velocity as is a change of speed. When a body has an acceleration, therefore, it may be undergoing a change of velocity of either kind, that is, its speed may be changing, or its direction of motion may be changing, or both. This is an important point and must not be forgotten: it is easy to overlook the fact that *an acceleration may be concerned with a change of direction of motion only,* and that accordingly a body may be given an acceleration without in any way affecting its speed.

These important facts may be more easily remembered if we state them in the following way:—

(*a*) An acceleration which is in the **same** direction as the direction of motion of a body will produce a change of **speed** only.

(*b*) An acceleration which is in a direction **perpendicular** to the direction of motion of a body will produce a change of **direction** only.

(*c*) An acceleration which is in any other direction will produce a change of both speed and direction.

We shall deal more fully with accelerations which produce changes of direction in the next chapter, when considering motion of rotation.

MEASUREMENT OF ACCELERATION. A body has unit acceleration when unit change of velocity is imparted to it in unit time. In the F.P.S. system of units, therefore, the systematic unit of acceleration is a change of velocity of one foot per second imparted in one second, that is, an acceleration of **one foot per second per second.** This unit requires careful consideration, and the student should make quite certain that he understands why the double *per second per second* is required.

Similarly, the systematic unit of acceleration in the C.G.S. system is an acceleration of **one centimetre per second per second.**

If a body moves with uniform acceleration, then the magnitude of its acceleration is equal to the change of velocity which it receives in each second.

If, on the other hand, the acceleration is varying, then, as with velocity, we have the choice of two methods of dealing with it, viz. :—

(*a*) We may determine the *actual* acceleration at any particular instant. This acceleration will, clearly, be different from that at the previous instant, and also from that at the following instant.

(*b*) We may find the *mean* or average acceleration of the body during some given interval of time. We can do this by dividing the total change of velocity of the body during this period by the total time occupied. For instance, if a body has received an increase of velocity of 44 feet per second in 8 seconds, then the mean acceleration is 44 feet per second divided by

8 seconds, i.e., it is equal to an acceleration of $5\frac{1}{2}$ feet per second per second. Clearly this result is independent of the *actual* acceleration at any particular instant, as to which, indeed, we have no information : in fact, the *mean* acceleration will be the same whether the body is moving with uniform or with variable acceleration, provided that the total change of velocity and the time occupied are the same in each case.

DIMENSIONS OF ACCELERATION. As we have seen above, we obtain the magnitude of an acceleration by dividing the change of velocity by the time occupied in making the change. Velocity is, as already shown, of the dimensions $[L][T]^{-1}$ and a period of time is of the dimensions $[T]$. Therefore the dimensions of any acceleration are $[L][T]^{-1}$ divided by $[T]$, that is, they are $[L][T]^{-2}$.

ACCELERATION DUE TO GRAVITY. When a body falls to the ground from a height, starting from rest, its velocity is initially zero but rapidly increases as the motion continues. Since the velocity is changing, the body is subject to acceleration. This acceleration has been frequently investigated experimentally and it is found to be approximately uniform.

The magnitude of the acceleration of a freely-falling body varies slightly at different parts of the surface of the earth, but it will be sufficiently accurate for all ordinary purposes if we assume it to be constant and equal to :—

32·2 feet per second per second (F.P.S. system), or **981 centimetres per second per second** (C.G.S. system).

This assumes that the resistance offered by the air to the falling of the body may be neglected, which is usually the case. The direction of the acceleration due to gravity may always be taken as vertical, although there are, in fact, slight divergences from this direction in some places.

The numerical value of the acceleration due to gravity,

whether expressed in F.P.S. or in C.G.S. units, is always denoted by the letter g.

ACCELERATION, VELOCITY, AND DISPLACEMENT. If we know the uniform acceleration acting upon a body, then we can very easily calculate the total change of velocity effected during any given period. Evidently it will be equal to the product of the acceleration and the time during which it acts. For example, an acceleration of 3 feet per second per second acting upon a body for 20 seconds will produce a change of velocity of $3 \times 20 = 60$ feet per second.

Similarly, if we know the mean value of a variable acceleration acting upon a body, we can find the total change of velocity produced in any given period, by multiplying the mean acceleration by the time during which it acts.

Again, if we know the uniform velocity of a body, or its mean velocity during a given period of time, then we can very easily calculate the distance it will move during that period, that is, its total change of position, or displacement. Clearly it will be equal to the product of the velocity and the time. For example, if a body moves with a mean velocity of 30 feet per second for 20 seconds, then the distance covered in that time will be $30 \times 20 = 600$ feet.

From these results it will be seen that, if we know the *uniform acceleration* acting upon a body during a given period of time, then we can not only calculate the *change of velocity* produced, but also the *distance covered*, provided that we know what was the initial velocity of the body, if any, at the commencement of the period.

As an example, let us take the case of a body moving from rest under an acceleration of, say, 4 feet per second per second. At the end of a period of 25 seconds the velocity attained will be $4 \times 25 = 100$ feet per second. The initial velocity was zero, so the mean velocity will be half the final velocity, that is, 50 feet per second. The

distance covered will be equal to the mean velocity multiplied by the time, that is, $50 \times 25 = 1{,}250$ feet.

Let us summarise the **relationship between displacement, velocity, and acceleration.**

(1) $\dfrac{\text{Change of position}}{\text{Time occupied}} = \text{Velocity (uniform or mean).}$

(2) $\dfrac{\text{Change of velocity}}{\text{Time occupied}} = \text{Acceleration (uniform or mean).}$

(3) Acceleration (uniform or mean) $\times$ Time = Change of Velocity.

(4) Velocity (uniform or mean) $\times$ Time = Change of Position.

These results are of great importance, and students should consider them very carefully and familiarise themselves, not only with the general statements, but also with their practical applications, as shown in the numerical examples given at the end of the chapter.

SYMBOLS AND FORMULÆ. We can, of course, express the above results more concisely if we use letters for the various quantities instead of words. We should only do this, however, when not to do so would cause difficulty or confusion. Otherwise we may be led into the fatal habit of regarding Mechanics as the art of manipulating various symbols and formulæ, of whose real meanings and significance we are only dimly aware. We shall do well, therefore, to avoid their use so far as that can be done without detriment to our work.

When we do find it necessary to employ abbreviations instead of the terms in full, then it will be wise to do so on some settled plan, using as far as possible symbols which either are the initial letters of the terms which they represent, or else have become so generally accepted in current use that they are not in danger of causing confusion. In any case, we must always take care to explain the meanings we give to the symbols we are about to employ before we commence to use them.

Neglect of this will probably make our work unintelligible to anyone but ourselves.

For dealing with problems in velocity, acceleration, etc., the following will be found the most convenient symbols :—

Let v = velocity.
a = acceleration.
g = acceleration due to gravity.
s = space or distance covered (= displacement).
t = time occupied.

Using these abbreviations, we see that the summary above can be expressed in the following way :—

$$(1)\ \frac{s}{t} = v$$

$$(2)\ \frac{v}{t} = a$$

$$(3)\ a.t = v$$

$$(4)\ v.t = s,$$

provided in each case that we are dealing with uniform velocity or acceleration ; or that, if the velocity or acceleration is variable, it is the mean value that we take.

Clearly, if the velocity of a body is changing uniformly, that is, if the acceleration is uniform, then the mean velocity will be the average of its initial and final velocities.

Let v_1 be the initial, and v_2 the final velocity.

Then the **mean velocity** will be $\frac{1}{2}(v_1 + v_2)$.

If the body starts from rest, then the initial velocity will be zero, and the mean velocity will accordingly be half the final velocity, i.e., mean velocity $= \frac{1}{2}v_2$. Also the final velocity will be the product of the acceleration and the time occupied, i.e., $v_2 = a.t$. Therefore we have :—

$$\text{Mean velocity} = \tfrac{1}{2}v_2 = \tfrac{1}{2}a.t.$$

Also the **displacement** (i.e., change of position, or space covered) = mean velocity $\times$ time occupied

$$= \tfrac{1}{2}at \times t = \tfrac{1}{2}at^2.$$

In the particular case of a **falling body,** that is, of a body moving under the attraction of the earth, we have :—

Velocity after time $t = gt$
Mean velocity during time $t = \frac{1}{2}gt$
Distance fallen in time $t = \frac{1}{2}gt^2$,

measuring the time from rest in each case.

GRAPHICAL METHODS. Many problems in connexion with velocity and acceleration can be dealt with conveniently by means of graphical constructions, such as **distance-time curves** and **velocity-time curves.** As it is our particular purpose in this book to deal with the *principles* of Mechanics, rather than to give an exhaustive account of *methods,* we shall not deal with these constructions here, but a brief account of some of them will be found in the Appendix.

Graphical methods are often of very great service, especially in avoiding lengthy and laborious calculations, and they should be carefully studied. Caution is, however, necessary in using them: they must never be employed as a means of solving problems without thought, merely by routine. Whatever the methods that we use we must take care to understand them, and we must ensure that we never allow methods to obscure principles. Results obtained graphically should be checked by some other means whenever possible.

SUMMARY OF CHAPTER 6

Speed is rate of motion, and has magnitude only. It is therefore a scalar quantity. **Velocity** is rate of change of position, and has both magnitude and direction. It is therefore a vector quantity. Velocities may be compounded and resolved by means of the **Parallelogram of Velocities.**

Relative velocity is the rate of change of position of one body with regard to another.

Velocity may be either **uniform** or **variable.** It is only uniform if both its magnitude and its direction are constant.

The systematic units of velocity are **one foot per second** in the F.P.S. system, and **one centimetre per second** in the C.G.S. system.

Acceleration is rate of change of velocity, and is a vector quantity. Accelerations may be compounded and resolved by means of the **Parallelogram of Accelerations.** Acceleration may be either **uniform** or **variable.**

Acceleration may be concerned with change of speed, or change of direction, or both. If a body is moving in a given direction, then an acceleration in the **same** direction will produce a change of **speed** only, an acceleration in a **perpendicular** direction will produce a change of **direction** only, and an acceleration in any other direction will produce a change of both speed and direction.

The systematic units of acceleration are **one foot per second per second** in the F.P.S. system, and **one centimetre per second per second** in the C.G.S. system.

The **acceleration due to gravity** is approximately uniform and equal to :—

32·2 feet per second per second,
or **981 centimetres per second per second.**

It is always denoted by **g.**

The **dimensions** of both velocity and speed are $[L][T]^{-1}$. The dimensions of acceleration are $[L][T]^{-2}$.

Acceleration (uniform or mean) × Time = Change of velocity.
Velocity (uniform or mean) × Time = Change of position.

EXAMPLES VI

(*For Hints on Working Examples, see page* 21.)

1. *A body moving with an initial velocity of* 30 *feet per second, receives an acceleration of* 3 *feet per second per second during* 7 *seconds. Determine its velocity at the end of the* 7 *seconds and the distance travelled in that time.*

Final velocity = initial velocity + change of velocity
= initial velocity + (acceleration × time)
= 30 feet per second + (3 feet per second per second × 7 seconds)
= 30 feet per second + 21 feet per second
= 51 feet per second.

Distance travelled
= mean velocity × time
= $\frac{1}{2}$(initial velocity + final velocity) × time
= $\frac{1}{2}$(30 feet per second + 51 feet per second) × 7 seconds
= $\frac{1}{2}$ × 81 feet per second × 7 seconds
= 283·5 feet.

2. *What acceleration must be given to a railway train in order that it may be moving with a velocity of* 30 *miles per hour two minutes after it starts from a station? How far will it have travelled in that time?*

Acceleration required

$$= \text{rate of change of velocity}$$
$$= \frac{\text{change of velocity}}{\text{time occupied}}$$
$$= \frac{30 \text{ miles per hour}}{2 \text{ minutes}}$$
$$= \frac{44 \text{ feet per second}}{120 \text{ seconds}}$$
$$= \cdot 367 \text{ foot per second per second.}$$

Distance travelled

$$= \text{average velocity} \times \text{time}$$
$$= \tfrac{1}{2} \text{ final velocity} \times \text{time}$$
$$= 15 \text{ miles per hour} \times \tfrac{1}{30} \text{ hour}$$
$$= \tfrac{1}{2} \text{ mile.}$$

3. *A man climbs the mast of a ship at the rate of* 120 *feet per minute, while the ship is moving ahead at* 9 *miles per hour, and a passenger is walking aft at* $2\frac{1}{2}$ *miles per hour. What is the relative velocity of the man* (*a*) *with regard to the sea, and* (*b*) *with regard to the passenger?*

$$\text{Speed of ship} = 9 \text{ miles per hour}$$
$$= 9 \times 88 \text{ feet per minute}$$
$$= 792 \text{ feet per minute.}$$

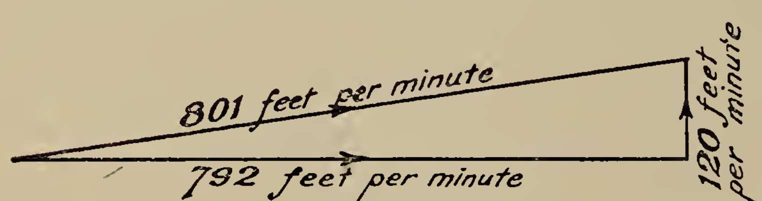


Fig. 55.

In one minute the man goes forward (with the ship) 792 feet. In the same time he goes upwards 120 feet. Therefore his velocity relative to the sea :—

$$= \sqrt{(792)^2 + (120)^2} \text{ feet per minute}$$
$$= \sqrt{627{,}000 + 14{,}400} \text{ feet per minute}$$
$$= \sqrt{641{,}400} \text{ feet per minute}$$
$$= 801 \text{ feet per minute.}$$

The direction of this velocity is at an angle with the horizontal :—

$$= \tan^{-1} \frac{120}{792}$$
$$= \tan^{-1} \cdot 1515$$
$$= 8^\circ \ 37'.$$

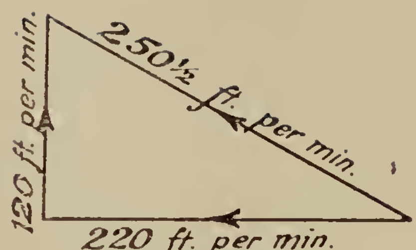


FIG. 56.

Speed of passenger $= 2\frac{1}{2}$ miles per hour
$= (2\frac{1}{2} \times 88)$ feet per minute
$= 220$ feet per minute.

Relative velocity of man compared with passenger

$$= \sqrt{(220)^2 + (120)^2}$$
$$= \sqrt{48{,}400 + 14{,}400}$$
$$= \sqrt{62{,}800}$$
$$= 250\tfrac{1}{2} \text{ feet per minute.}$$

The direction of this velocity is at an angle with the horizontal :—

$$= \tan^{-1} \frac{120}{220}$$
$$= \tan^{-1} \cdot 545$$
$$= 28^\circ\ 36'.$$

All these results can be obtained graphically, by drawing the parallelogram of velocities in each case, as indicated above (see Figs. 55 and 56).

4. *Find the distance moved in five minutes by a railway train which starts from rest and moves with an acceleration of* ·35 *foot per second per second for* 3 *minutes and after that with uniform speed.*

Greatest speed attained

$= \text{acceleration (uniform)} \times \text{time}$
$= \cdot 35$ foot per second per second $\times$ 180 seconds
$= 63$ feet per second
$$= \frac{63 \times 3{,}600}{5{,}280} \text{ miles per hour}$$
$= 43$ miles per hour.

During the first three minutes the *average* speed is therefore $\frac{1}{2} \times 43 = 21\frac{1}{2}$ miles per hour.

$\therefore$ Distance travelled in first three minutes

$= \text{mean speed} \times \text{time}$
$= 21\frac{1}{2}$ miles per hour $\times \frac{1}{20}$ hour
$= 1{\cdot}075$ miles.

During the remaining two minutes the distance travelled

$$= \text{uniform speed} \times \text{time}$$
$$= 43 \text{ miles per hour} \times \tfrac{1}{30} \text{ hour}$$
$$= 1{\cdot}432 \text{ miles.}$$

$\therefore$ Total distance travelled from rest in five minutes

$$= 1{\cdot}075 \text{ miles} + 1{\cdot}432 \text{ miles}$$
$$= 2{\cdot}507 \text{ miles.}$$

5. *How long will it take for a stone to drop from the top of a cliff to the sea which is* 150 *feet below it?*

Velocity of stone on reaching sea

$$= \text{acceleration due to gravity} \times \text{time occupied in falling}$$
$$= 32{\cdot}2 \text{ feet per second per second} \times t \text{ seconds}$$
$$= 32{\cdot}2.t \text{ feet per second.}$$

Mean velocity of stone during fall

$$= \tfrac{1}{2}(\text{initial velocity} + \text{final velocity})$$
$$= \tfrac{1}{2}(0 + 32{\cdot}2.t) \text{ feet per second}$$
$$= 16{\cdot}1.t \text{ feet per second.}$$

$$\text{Distance fallen} = \text{mean velocity} \times \text{time}$$
$$= 16{\cdot}1.t \times t$$
$$= 16{\cdot}1.t^2 \text{ feet,}$$

whence, as the distance fallen is 150 feet, we have :—

$$16{\cdot}1.t^2 = 150 \text{ feet}$$
$$t^2 = \frac{150}{16{\cdot}1} \text{ sec.}^2$$
$$= 9{\cdot}32 \text{ sec.}^2,$$

so that $t = \sqrt{9{\cdot}32}$

$$= 3{\cdot}05 \text{ seconds.}$$

Actually the time occupied would be slightly longer, owing to the resistance of the air.

6. *A motor-car is travelling at a velocity of* 25 *miles per hour. What acceleration must be given to it in order to bring it to rest* (*a*) *in* 10 *seconds,* (*b*) *in* 100 *feet?*

(*a*) Acceleration required

$$= \text{rate of change of velocity}$$
$$= \frac{\text{change of velocity}}{\text{time occupied}}$$

$$= \frac{-25 \text{ miles per hour}}{10 \text{ seconds}}$$

(the change of velocity is *negative* because it is a reduction of speed)

$$= \frac{-25 \times 22}{15 \times 10} \text{ feet per second per second}$$

(since 15 miles per hour = 22 feet per second)

$$= -3\tfrac{2}{3} \text{ feet per second per second.}$$

(*b*) Time occupied

$$= \frac{\text{distance travelled}}{\text{mean velocity}}$$

$$= \frac{100 \text{ feet}}{\frac{1}{2} \times 25 \text{ m.p.h.}}$$

$$= \frac{2 \times 100 \times 15}{25 \times 22} \text{ seconds}$$

$$= 5{\cdot}45 \text{ seconds.}$$

Acceleration required

$$= \frac{\text{change of velocity}}{\text{time occupied}}$$

$$= \frac{-25 \text{ miles per hour}}{5{\cdot}45 \text{ seconds}}$$

$$= \frac{-25 \times 22}{15 \times 5{\cdot}45} \text{ feet per second per second}$$

$$= -6{\cdot}72 \text{ feet per second per second}$$

(the minus sign indicating a *retardation*).

7. Find the velocity acquired in 3 minutes by a body which moves from rest with an acceleration of ·37 foot per second per second.

8. A ship sails in such a way that it moves Westwards at 5·7 knots, and Southwards at 6·3 knots, at the same time. Find its velocity.

9. Find the component velocities Northwards and Eastwards of a train travelling East-North-East at 48 miles per hour.

10. A body moving with an initial velocity of 7 feet per second, receives an acceleration of 2 inches per second per second. Determine its velocity (*a*) after 5 seconds, and (*b*) after $4\frac{1}{2}$ minutes. How far will it travel in the fifth minute from the commencement of the acceleration ?

11. What must be the acceleration of a motor-car in order that

it may be moving with a velocity of 23 miles per hour $1\frac{3}{4}$ minutes after it starts from rest ?

12. Find the real velocity of a train which appears to be moving with a speed of 47 miles per hour due West when seen from a train which is travelling South-East at 15 miles per hour.

13. A swimmer crosses a river, 126 feet in width, by swimming directly towards the opposite bank at a speed of $1\frac{1}{4}$ miles per hour. If, at the same time, the water is flowing at a mean speed of 33 yards per minute, how long will it take the swimmer to cross the river, and how far down stream will he be carried ? Determine also his velocity with regard to a point on the bank.

14. Find how far a railway train will travel in 6 minutes, if it starts from rest and moves with an acceleration of 3 inches per second per second until it has attained a speed of 40 miles per hour and then moves with uniform speed.

15. A stone falls vertically through a distance of 274 feet : find its velocity on reaching the ground, and the time occupied in falling.

16. A railway train starts from rest and moves with an acceleration of $\frac{3}{4}$ foot per second per second. Ten seconds later another train starts from the same place and travels in the same direction with an acceleration of $1\frac{1}{4}$ feet per second per second. Find how long it will take for the second train to overtake the first, and how far they will have travelled from the starting point.

17. A passenger walks along the corridor of an express train at a speed of $2\frac{3}{4}$ miles per hour in the opposite direction to that in which the train is travelling. If the train crosses a river in which the water is flowing at 2 miles per hour, at right angles, and is then going at 53 miles per hour, find the velocity of the water relative to the passenger.

18. A body is projected horizontally from the top of a tower 300 feet in height, with a speed of 52 feet per second. Find how long it will take to reach the ground, and how far from the foot of the tower it will fall.

19. A bullet is fired horizontally with an initial velocity of 1,100 feet per second, at a height of 4 feet 6 inches above the ground. Neglecting air resistance, determine how far it will travel before it strikes the ground.

20. A man walking Northwards at $4\frac{1}{4}$ miles per hour feels the wind blowing apparently due East at $6\frac{1}{2}$ miles per hour. Find the true velocity of the wind.

21. The current in a river 60 yards wide is flowing at 50 yards per minute. A man wishes to cross the river, at right angles, in a boat which he can pull at $2\frac{1}{2}$ miles per hour in still water. Find in

which direction he must head the boat, and how long it will take him to cross.

22. A man running appears to an observer who is due West of him to be moving Southwards at $5\frac{3}{4}$ miles per hour. To another observer who is due North of him, the runner appears to be moving Eastwards at $3\frac{1}{2}$ miles per hour. What is the true velocity of the runner ?

23. A stone is dropped from the top of a tower 288 feet in height, at the same instant as another stone is thrown vertically upwards from the ground with a velocity of a mile a minute. Find how far the second stone will rise, and which stone will reach the ground first and by how long.

24. A ball is dropped from a height of 455 feet. With what velocity must another ball be thrown downwards a second later in order to reach the ground a second sooner than the first ball ?

25. A train 650 feet in length is half-a-mile from a signal-cabin and travelling at 70 miles per hour, when the brakes are applied, giving a retardation of 1·5 feet per second per second. Find the velocity of the train when its engine passes the cabin and when the brake-van passes the cabin, and the time occupied by the whole train in passing the cabin.

26. A body starts from rest at a point A and moves with an acceleration of 1,350 feet per minute per minute towards a point B which is 1·6 miles from A. If another body starts from B at the same instant with an initial velocity of 80 miles per hour and a retardation of one F.P.S. unit, and moves towards A, find how far apart the bodies will be when their velocities are equal, and at what point they will pass each other.

27. A stone is thrown upwards from the ground with a velocity of 85 feet per second at an angle of 60 degrees from the horizontal. Find how long it will take the stone to reach the ground again and how far away it will fall.

CHAPTER 7: ROTATION

WE have seen that when a body is moving we can classify its movement as being one of **translation,** or of **rotation,** or of the two combined.

We may remind ourselves that a body has motion of *translation* when it moves as a whole from some position A along a straight line to some other position B (Fig. 57): it has motion of *rotation* when it does not change its position from one place to another, but remaining at some place A, turns round and round about some fixed axis.

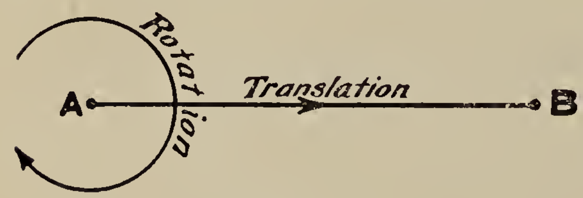


FIG. 57.

In considering velocity and acceleration we have, up to the present, only dealt with motion of translation, that is, motion along a line, such as AB. To be precise, therefore, we should speak of such velocity and acceleration as **linear velocity** and **linear acceleration.**

We may define the *linear velocity* of a body as its *rate of translation,* that is, the rate at which it moves from one position A to some other position B. The *linear acceleration* of a body we may define as the rate of change of its linear velocity.

ANGULAR VELOCITY AND ACCELERATION. We now have to consider the other kind of motion, that is, motion of rotation, and in this case we have to deal with angular velocity and angular acceleration.

We may define the **angular velocity** of a body as its *rate of rotation,* that is, the rate at which it turns round and round about some fixed axis.

The **angular acceleration** of a body we may define as *the rate of change of its angular velocity.*

Like linear velocity, angular velocity may be either

uniform or **variable.** If it is uniform, then equal *angles* will be turned through in equal intervals of time : if it is variable, then unequal angles will be turned through in equal intervals of time.

Similarly, angular acceleration may be either **uniform** or **variable.** In the former case equal changes of angular velocity will take place in equal intervals of time : in the latter case unequal changes of angular velocity will take place in equal intervals of time.

VECTOR QUANTITIES OF THE SECOND CLASS. Angular velocity and angular acceleration are **vector quantities.** This may not be evident at first sight, but a little consideration will show the correctness of the statement.

We can represent the *direction* of rotation by a straight line AB drawn parallel to the *axis* of rotation.

We can represent the *sense* of the rotation by an arrow-head marked on the line AB, so that, looking along the line in the direction of the arrow, if the rotation appears to be clockwise we call it negative, and if it appears to be counter-clockwise we call it positive (see Fig. 58).

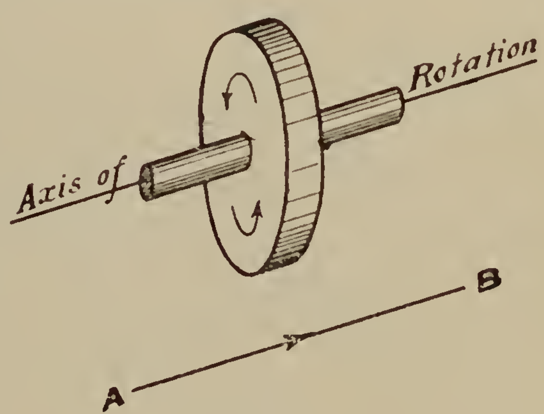


Fig. 58.

We can represent the *magnitude* of the angular velocity, or the angular acceleration, as the case may be, by the length of the line AB.

If, then, angular velocity and angular acceleration are vector quantities, they must obey the Parallelogram Law, and it can be proved that they do so. We will not consider the proof here, but will content ourselves with an illustration.

Suppose a fly-wheel is held in a frame so that it can rotate freely about a horizontal axis AA, as shown in Fig. 59. Then if the frame is kept at rest, and the fly-wheel caused to rotate, we can represent its angular

velocity by the line O*a* drawn parallel to the axis of rotation AA.

Now if the rotation of the fly-wheel about AA is stopped, and the frame and fly-wheel together are caused to rotate about the vertical axis BB, then we can represent the angular velocity of the fly-wheel by a vertical line, i.e., by a line O*b* drawn parallel to the new axis of rotation BB.

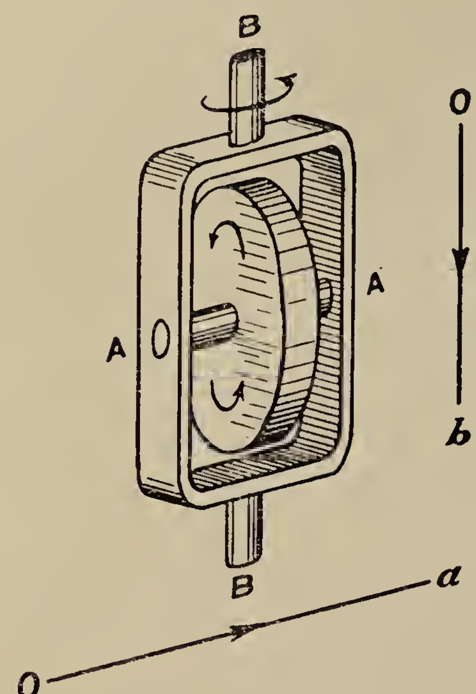


Fig. 59.

Imagine, now, that we make a fresh start and again set the wheel spinning about its own axis AA. If, while it is spinning, we rotate the whole apparatus about the vertical axis BB, then evidently the wheel has now two different motions of rotation at the same time. It is rotating about the horizontal axis AA and is also rotating about the vertical axis BB. We can find the combined effect of these two motions at any instant by means of the **Parallelogram of Angular Velocities.**

We draw O*a* parallel to the horizontal axis AA about which the wheel is rotating at the given instant, to represent the angular velocity of the wheel about AA. From the same point O we draw O*b* parallel to the vertical axis BB to represent the angular velocity of the wheel about BB. Then if we complete the parallelogram (Fig. 60), the diagonal O*c*, drawn through the same point O, represents the **resultant angular velocity** of the wheel at the given instant, its direction being parallel to the axis of the resultant angular velocity, and its length giving the magnitude of that velocity.

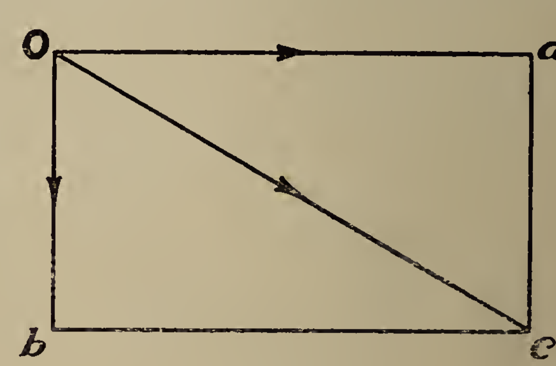


Fig. 60.

Students who find difficulty in grasping the signifi-

cance of this result may defer its consideration until later. They should, however, notice the important fact that such quantities as angular velocity and angular acceleration are vector quantities—the direction of the vector being in each case that of the *axis* of rotation —and that being vector quantities they are therefore compounded and resolved in accordance with the Parallelogram Law.

Obviously, there are differences between vector quantities of this kind and those with which we have previously dealt. We may mark this difference by calling these new vector quantities **vector quantities of the second class,** while those connected with linear motion we term **vector quantities of the first class,** i.e. :—

Vector quantities of the *first* class are those connected with motion of *translation*: vector quantities of the *second* class are those connected with motion of *rotation*.

MEASUREMENT OF ANGULAR VELOCITY. A body has unit angular velocity when it turns through unit angle in unit time. In either the F.P.S. system or the C.G.S., the systematic unit angle is the *radian* and the systematic unit of time is the *second*: consequently the systematic unit of angular velocity in either system is a velocity of **one radian per second,** and this is the unit we shall normally employ.

In practical questions connected with engineering, etc., it is more convenient to take as our unit angle an angle of 2π radians or 360 degrees, that is, one complete *revolution*. Also, we may, for such questions, find it preferable to use the *minute* as our unit of time. The practical unit of angular velocity will then be a velocity of **one revolution per minute.** This unit, although more convenient for practical purposes, is not suitable for many of the theoretical questions which we have to study in Mechanics. It is the angular velocity of a body which turns completely round once in each minute.

We can very easily convert an angular velocity expressed in systematic units into the equivalent velocity

in practical units or vice versa. In one complete revolution a body turns through an angle of 2π radians: therefore a velocity of one revolution per minute is equal to a velocity of 2π radians per minute, that is, to a velocity of $2\pi/60 = \pi/30$ radians per second. To convert from revolutions per minute to radians per second, therefore, all that we have to do is to multiply by π and divide by 30. Similarly, to convert an angular velocity from radians per second to revolutions per minute, all that we have to do is to multiply by 30 and divide by π.

DIMENSIONS OF ANGULAR VELOCITY. We obtain the magnitude of an angular velocity by dividing the angle turned through (in radians) by the time occupied (in seconds). The angle in radians is measured by the ratio *arc/radius* and its dimensions are therefore **[L]/[L]**, that is $\mathbf{[L]^0}$ or **1**, which shows that a radian is a purely *numerical* quantity. The dimensions of a period of time are one dimension of time, that is **[T]**. Therefore the dimensions of an angular velocity are **1** divided by **[T]**, that is $\mathbf{[T]^{-1}}$.

Angular velocity in radians per second is usually denoted by the Greek letter ω (ōmĕga), and angular velocity in revolutions per minute by the letter **N**.

ANGULAR VELOCITY IN TERMS OF LINEAR SPEED. We frequently require to express the angular velocity of a body in terms of the linear speed of some point on the body, or vice versa. This we can very readily do.

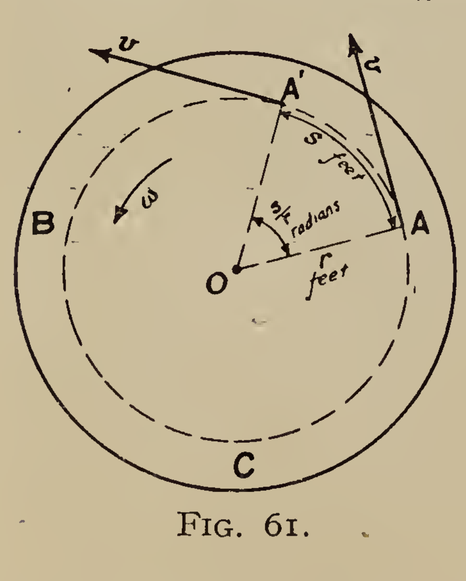


Fig. 61.

Suppose that a rigid body ABC (Fig. 61) is rotating about a fixed axis through O in the figure (perpendicular to the plane of the paper), with uniform angular velocity ω radians per second. Then any point A in the body, at a distance r feet from the axis, will move round the axis in a circular path of radius r, as the body rotates.

Let the speed at which it moves along this circular path be v feet per second. When the point has moved through a distance s feet along the circle, the body will have turned through an angle equal to $\frac{s}{r}$ radians. The time occupied will be $\frac{s}{v}$ seconds. Therefore the angular velocity of the body will be :—

$$\omega = \frac{\text{angle turned through}}{\text{time occupied}} = \frac{s}{r} \div \frac{s}{v} = \frac{v}{r} \text{ radians per second.}$$

We see therefore that the angular velocity of a body may be obtained by dividing the linear speed of any point of the body by the distance of that point from the axis of rotation.

We can confirm this by considering the dimensions of the quantities involved. The dimensions of a linear speed are, as we have seen, $[L][T]^{-1}$. The dimensions of the radius of rotation are $[L]$, since it is a length. Therefore if we divide linear speed by radius of rotation, we obtain a quantity of which the dimensions are $[L][T]^{-1}$ divided by $[L]$, that is $[T]^{-1}$, which we have already found to be the dimensions of an angular velocity.

LINEAR SPEED IN TERMS OF ANGULAR VELOCITY. Conversely, the linear speed of any point of a rotating body can be obtained by multiplying the angular velocity of the body by the perpendicular distance of the point from the axis of rotation, i.e.,

$$v = \omega . r.$$

It should be noticed that *the angular velocity of every part of a rigid body must be the same*: otherwise part of it would turn through a complete revolution while another part had only turned through part of a revolution, which is clearly impossible.

Also, from the above work it follows that the farther any point of a body is from the axis of rotation the

greater is its linear speed. In a fly-wheel, for example, a point on the rim will obviously move faster than a point on the boss, for it has to make a complete revolution round a large circle in exactly the same time that the latter point has to make a revolution round a small circle.

MEASUREMENT OF ANGULAR ACCELERATION. A body has unit angular acceleration when it receives unit change of angular velocity in unit time. In either the F.P.S. system or the C.G.S., the systematic unit change of angular velocity is a change of one radian per second, and the systematic unit of time is the second : consequently the systematic unit of angular acceleration, in either system, is an acceleration of **one radian per second per second,** and this is the most convenient unit for use in the study of Mechanics.

The corresponding practical unit is an angular acceleration of **one revolution per minute per minute,** but this is not very much used.

Angular acceleration in radians per second per second may be denoted by the Greek letter α (alpha), and angular acceleration in revolutions per minute per minute by the letter **A.**

DIMENSIONS OF ANGULAR ACCELERATION. We obtain the angular acceleration of a body by dividing the change of angular velocity by the time occupied in making that change. The dimensions of angular velocity are, as we have seen, $[T]^{-1}$, and the dimensions of a period of time are $[T]$, so that the dimensions of an angular acceleration are $[T]^{-1}$ divided by $[T]$, that is $[T]^{-2}$.

ANGULAR ACCELERATION IN TERMS OF LINEAR ACCELERATION. We can express the angular acceleration of a body in terms of the linear acceleration of any point in the body, and vice versa.

Let α = angular acceleration in radians per second per second,

ω_1 = initial angular velocity in radians per second,

and ω_2 = angular velocity in radians per second after t seconds.

Then the change of angular velocity in t seconds is $(\omega_2 - \omega_1)$ radians per second, and the angular acceleration (uniform or mean) is equal to the change of angular velocity divided by the time occupied, i.e.,

$$\alpha = \frac{\omega_2 - \omega_1}{t} \text{ radians per second per second.}$$

Now ω_1 is equal to $\frac{v_1}{r}$, and ω_2 to $\frac{v_2}{r}$, where

v_1 = initial linear speed, in feet per second, of any point in the body distant r feet from the axis of rotation, and

v_2 = linear speed, in feet per second, of the same point after t seconds.

Therefore we have :—

$$\alpha = \frac{v_2 - v_1}{r.t}$$

$$= \frac{a}{r} \text{ radians per second per second,}$$

where a = linear acceleration of the point in feet per second per second.

We can confirm this by considering the dimensions of the quantities involved. The dimensions of a linear acceleration are, as we have already seen, $[L][T]^{-2}$. The dimensions of the radius of rotation, which is a length, are $[L]$. Therefore the dimensions of an angular acceleration, obtained by dividing a linear acceleration by a radius, are $[L][T]^{-2}$ divided by $[L]$, that is $[T]^{-2}$, which is the same result as we have already obtained by another method.

Conversely, the linear acceleration of any point of a rotating body can be obtained by multiplying the angular acceleration of the body by the distance of the point from the axis of rotation, i.e.,

$$a = \alpha.r \text{ feet per second per second.}$$

NORMAL ACCELERATION. Let us now consider the motion of a body which moves along a curved path with uniform speed, such, for example, as a railway train. The *speed* is uniform but the *velocity* is not, for, as we have already noticed more than once, the term velocity includes the two ideas of *speed* and *direction,* and in this case only one of these is constant.

Therefore we see that because the direction of motion is varying, the velocity of the body is varying. In the case of the train, at some particular instant the velocity may be, for example, due South. As the train travels round the curve the direction is continually changing, and a little later the velocity may be South-West in direction.

It follows that *whenever a body travels along any curved path it must have an acceleration,* for the velocity of the body is changing and acceleration is simply rate of change of velocity. Let us now repeat certain statements which we have already made: reiteration is justifiable, for they are very important.

(*a*) An acceleration which is in the **same** direction as the direction of motion of a body, will produce a change of **speed** only.

(*b*) An acceleration which is in a direction **perpendicular** to the direction of motion of a body, will produce a change of **direction** only.

(*c*) An acceleration which is in any other direction will produce a change of both speed and direction.

In view of the work which we have already done in connexion with acceleration, (*a*) and (*b*) should be almost self-evident. In case (*c*) we can resolve the acceleration into components along and perpendicular to the direction of motion of the body. The component in the same direction as the direction of motion of the body will then produce a change of speed, while the other component will produce a change of direction.

We now have to study case (*b*) in more detail.

An acceleration that does not produce a change of speed but only a change of direction of motion, is known

as a **normal acceleration,** because of the fact that it always acts in a direction which is *normal* (i.e., perpendicular) to the direction in which the body is moving at any particular instant.

Normal acceleration being, like any other acceleration, the *rate* of change of velocity, it follows that the less the time occupied in making a given change of velocity, the greater must be the acceleration. Clearly, therefore, the sharper the curve the quicker the change of velocity will take place and the greater will be the normal acceleration, whatever may be the constant speed during the change of direction.

We may, of course, as already noticed, meet with a case where both the speed and the direction of motion are changing. In such a case the body will have two accelerations: one along its path producing change of speed, and one normal to its path producing change of direction. We can, if we wish, find the resultant acceleration at any instant by means of the Parallelogram of Accelerations.

CIRCULAR MOTION. The path of a moving body may be a straight line or it may be any kind of curve: the simplest and most important curve is the *circle*. Let us therefore determine the magnitude of the normal acceleration of a body which is moving along a circular path of radius r feet, at a constant speed of v feet per second.

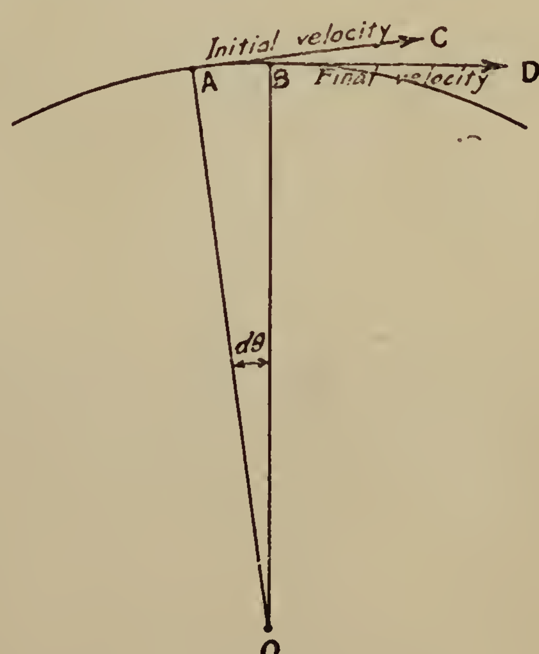


FIG. 62.

Suppose that A and B (Fig. 62) are two points on the circular path very close together, and that O is the centre of the circle, so that $OA = OB = r$. Let the very small angle AOB $= d\theta$.

When the body is at A its velocity will be, for the

instant, v feet per second in the direction AC perpendicular to OA. When it is at B its velocity will be v feet per second in the direction BD perpendicular to OB.

The *final* velocity must, of course, be the resultant of the *initial* velocity and the *change* of velocity; so that if we represent the initial velocity by the vector EF (Fig. 63), drawn parallel to AC, and the final velocity by the vector EG, drawn parallel to BD from the same point E, then the change of velocity will be represented by the remaining side FG of the triangle, as shown. (Notice that this triangle is really the Parallelogram of Velocities in an abbreviated form, as explained on page 31, in connexion with the Parallelogram of Forces). Clearly the angle FEG is equal to the angle AOB which is the angle $d\theta$.

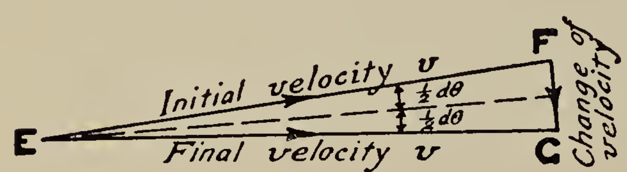


FIG. 63.

We have then :—

$$\text{Change of velocity} = FG$$
$$= 2.EF \sin \tfrac{1}{2}d\theta$$
$$= 2.v.\sin \tfrac{1}{2}d\theta \text{ feet per second.}$$

$$\text{Normal acceleration} = \text{rate of change of velocity}$$
$$= \frac{\text{change of velocity}}{\text{time occupied}}$$

(provided that the angle turned through is indefinitely small and that the change of velocity and the time occupied are therefore indefinitely small also),

$$= \frac{2.v.\sin \tfrac{1}{2}d\theta}{t} \text{ feet per sec. per sec.}$$

But $d\theta$ is an indefinitely small angle, so that $\sin \tfrac{1}{2}d\theta = \tfrac{1}{2}d\theta$: also $t =$ arc AB divided by v (i.e., time occupied = distance travelled divided by speed).

Hence :—

$$\text{Normal acceleration} = \frac{2.v.\frac{1}{2}d\theta}{AB/v}$$

$$= \frac{v^2.d\theta}{AB}$$

$$= \frac{v^2}{r} \text{ feet per second per second,}$$

since $d\theta = AB/r$.

We see then that the normal acceleration of a body moving along a circular path is equal to the square of the linear speed divided by the radius of the circle, and its direction is clearly perpendicular to the indefinitely small arc AB, i.e., it is directed towards the centre of the circle.

It should be noticed that normal acceleration is a *variable* acceleration, for although its magnitude is constant (for a body moving round a circle with uniform speed), its direction is always changing, being along the radius from the body to the centre of the circle.

Since v is in feet per second, and r is in feet, therefore the normal acceleration, v^2/r, is in terms of :—

$$\frac{\text{feet per second} \times \text{feet per second}}{\text{feet}},$$

that is, it is in *feet per second per second*, the same as any other linear acceleration.

Also since $v = \omega r$

therefore $v^2 = \omega^2 r^2$

and normal acceleration $= \frac{v^2}{r}$

$$= \frac{\omega^2 r^2}{r}$$

$$= \omega^2 r,$$

i.e., the normal acceleration is equal to the square of the angular velocity multiplied by the radius of the circle.

DIMENSIONS OF NORMAL ACCELERATION. Taking the expression for normal acceleration as v^2/r, the dimensions will be $\{[L][T]^{-1}\}^2 \div [L]$, that is $[L][T]^{-2}$, which are the dimensions of a linear acceleration, as we have already seen.

Taking the alternative expression, that is, $\omega^2 r$, the dimensions will be $\{[T]^{-1}\}^2 \times [L]$, which gives us $[L][T]^{-2}$, as before.

SUMMARY OF CHAPTER 7

Angular velocity is rate of rotation. **Angular acceleration** is rate of change of angular velocity. Both are vector quantities and therefore obey the Parallelogram Law.

Vector quantities of the **first class** are those connected with **linear motion.** Vector quantities of the **second class** are those connected with **angular motion.**

The systematic unit of angular velocity is **one radian per second,** in either the F.P.S. or the C.G.S. system. The practical unit is **one revolution per minute.**

The dimensions of angular velocity are $[T]^{-1}$.

If ω is the angular velocity of a body in radians per second, and v is the linear speed of any point in the body at a distance r from the axis of rotation, then $\omega = v/r$, and $v = \omega.r$.

The systematic unit of angular acceleration is **one radian per second per second.** The practical unit is **one revolution per minute per minute,** but this is not much used.

The dimensions of angular acceleration are $[T]^{-2}$.

If α = angular acceleration in radians per second per second, a = linear acceleration in feet per second per second, and r = radius of rotation in feet, then $\alpha = a/r$, and $a = \alpha.r$.

An acceleration which does not produce a change of speed but only a change of direction of motion, is known as a **normal acceleration,** because its direction is necessarily perpendicular to the direction of motion at any instant.

In the case of **uniform circular motion,** that is, motion where the path of the moving body is a circle and the speed is uniform, the magnitude of the normal acceleration is v^2/r feet per second per second, which is equal to $\omega^2.r$ feet per second per second.

Normal acceleration is a particular case of linear acceleration : its dimensions are therefore $[L][T]^{-2}$.

EXAMPLES VII

(*For Hints on Working Examples, see page* 21.)

1. *A wheel,* 4 *feet* 6 *inches in diameter, makes* 300 *revolutions per minute. Determine* (*a*) *its angular velocity in radians per second, and* (*b*) *the linear speed of a point on the rim of the wheel.*

(*a*) Angular velocity of wheel

$$= 300 \text{ revolutions per minute}$$
$$= 5 \text{ revolutions per second}$$
$$= (5 \times 2\pi) \text{ radians per second}$$
$$= 31{\cdot}42 \text{ radians per second.}$$

(*b*) Linear speed of point on rim of wheel

$$= \text{angular velocity} \times \text{radius}$$
$$= 31{\cdot}42 \text{ radians per second} \times 2\tfrac{1}{4} \text{ feet}$$
$$= 70{\cdot}7 \text{ feet per second.}$$

2. *The linear speed of a point on the rim of a fly-wheel is* 5,400 *feet per minute. If the speed falls in* 3·5 *seconds to* 5,000 *feet per minute, and the diameter of the wheel is* 5 *feet, what is the angular acceleration?*

Change of speed of rim in $3\frac{1}{2}$ seconds

$$= 5{,}400 \text{ feet per minute} - 5{,}000 \text{ feet per minute}$$
$$= 400 \text{ feet per minute}$$
$$= \frac{400}{60} \text{ feet per second}$$
$$= 6\tfrac{2}{3} \text{ feet per second.}$$

Rate of change of rim speed (linear acceleration)

$$= \frac{\text{change of speed}}{\text{time occupied}}$$
$$= \frac{6\frac{2}{3} \text{ feet per second}}{3{\cdot}5 \text{ seconds}}$$
$$= 1{\cdot}905 \text{ feet per second per second.}$$

$\therefore$ Rate of change of angular velocity (angular acceleration)

$$= \frac{\text{linear acceleration}}{\text{radius}}$$
$$= \frac{1{\cdot}905 \text{ feet per second per second}}{2\frac{1}{2} \text{ feet}}$$
$$= {\cdot}762 \text{ radian per second per second.}$$

$$= \frac{\cdot 762 \times 60 \times 60}{2\pi} \text{ revolutions per minute per minute}$$

$$= 437 \text{ revolutions per minute per minute.}$$

This is, of course, a retardation.

3. *A body moves from rest along a circular path of* 3 *feet radius, with an angular acceleration of* 42 *revolutions per minute per minute. Find its linear speed and also its resultant acceleration* (*a*) *after* 6 *seconds, and* (*b*) *after* 10 *seconds.*

(*a*) Angular velocity after 6 seconds

$$= 42 \text{ revolutions per minute per minute} \times \tfrac{6}{60} \text{ minute}$$
$$= 4\cdot 2 \text{ revolutions per minute}$$
$$= \frac{4\cdot 2 \times 2\pi}{60} \text{ radians per second}$$
$$= \cdot 44 \text{ radian per second.}$$

$\therefore$ Linear speed after 6 seconds

$$= \text{angular velocity} \times \text{radius}$$
$$= \cdot 44 \text{ radian per second} \times 3 \text{ feet}$$
$$= 1\cdot 32 \text{ feet per second.}$$

Normal acceleration after 6 seconds

$$= \frac{(\text{linear speed})^2}{\text{radius}}$$
$$= \frac{(1\cdot 32 \text{ feet per sec.})^2}{3 \text{ feet}}$$
$$= \cdot 58 \text{ foot per second per second.}$$

Constant angular acceleration

$$= 42 \text{ revolutions per minute per minute}$$
$$= \frac{42 \times 2\pi}{60 \times 60} \text{ radians per second per second}$$
$$= \cdot 0733 \text{ radian per second per second.}$$

$\therefore$ Linear acceleration in direction of motion (constant magnitude)

$$= \text{angular acceleration} \times \text{radius}$$
$$= \cdot 0733 \text{ radian per second per second} \times 3 \text{ feet}$$
$$= \cdot 22 \text{ foot per second per second.}$$

Therefore after 6 seconds the body is moving with a linear acceleration of ·22 foot per second per second in the direction of motion and a normal acceleration of ·58 foot per second per second. Its resultant

acceleration at that instant is therefore found by compounding together these two accelerations by means of the Parallelogram of Accelerations, as shown (Fig. 64).

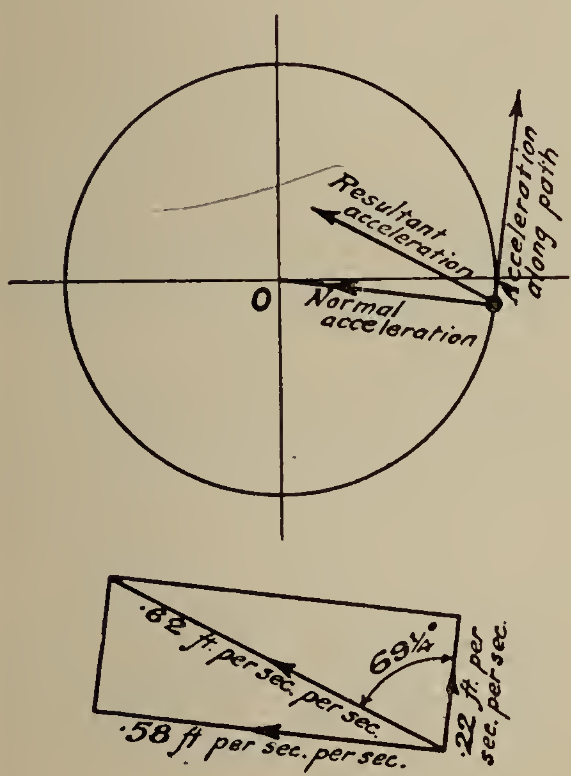


FIG. 64.

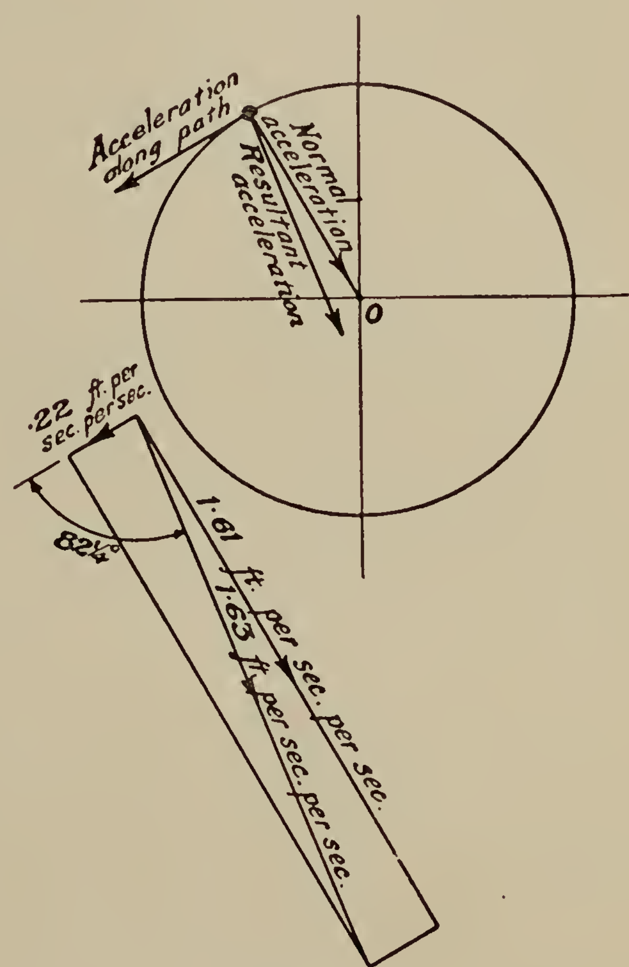


FIG. 65.

Drawing the parallelogram to as large a scale as possible, and measuring the resultant, as represented by the diagonal, we find that its magnitude is ·62 foot per second per second and that its direction makes an angle of $69\frac{1}{4}$ degrees with the direction of motion at that instant.

(*b*) Angular velocity after 10 seconds

$$= 42 \text{ revolutions per minute per minute} \times \tfrac{10}{60} \text{ minute}$$
$$= 7 \text{ revolutions per minute}$$
$$= \frac{7 \times 2\pi}{60} \text{ radians per second}$$
$$= \cdot 733 \text{ radian per second.}$$

∴ Linear speed after 10 seconds

$$= \text{angular velocity} \times \text{radius}$$
$$= \cdot 733 \text{ radian per second} \times 3 \text{ feet}$$
$$= 2\cdot 2 \text{ feet per second.}$$

Normal acceleration after 10 seconds

$$= \frac{(\text{linear speed})^2}{\text{radius}}$$

$$= \frac{(2{\cdot}2 \text{ feet per second})^2}{3 \text{ feet}}$$

$$= 1{\cdot}61 \text{ feet per second per second.}$$

Therefore, after 10 seconds the body is moving with a linear acceleration of ·22 foot per second per second in the direction of motion and a normal acceleration of 1·61 feet per second per second. As before, we find the resultant acceleration at that instant by compounding together these two accelerations by means of the Parallelogram of Accelerations (Fig. 65).

Measuring the diagonal of the parallelogram, we find that the magnitude of the resultant acceleration is 1·63 feet per second per second and that its direction makes an angle of $82\frac{1}{4}$ degrees with the direction of motion at that instant.

4. A wheel, 1·48 metres in diameter, makes 280 revolutions per minute. Find (i) its angular velocity in radians per second, and (ii) the linear speed, in C.G.S. units, of a point on the rim of the wheel.

5. The linear speed of a point on the rim of a wheel is a mile a minute. If the diameter of the wheel is 88 inches, find the angular velocity of the wheel (i) in C.G.S. units, and (ii) in practical units.

6. Determine the diameter of a fly-wheel so that when running at 400 revolutions per minute its rim speed shall not exceed 96 F.P.S. units.

7. Find the angular velocity, in practical units, of a wheel 140 centimetres in diameter, if its rim speed is 90 kilometres per hour.

8. What is the angular velocity (in systematic units) of a railway train when rounding a curve of 3 furlongs radius at a speed of 27 miles per hour ?

9. The driving wheels of a locomotive are 6 feet 6 inches in diameter. Find their angular velocity when the engine is travelling at 70 miles per hour.

10. The wheels of a cart are 52 inches in diameter and make 16 revolutions per minute. At what speed is the cart travelling ?

11. A point on a revolving body is at a distance of 2 feet $8\frac{1}{2}$ inches from the axis of rotation, and its linear speed is 4,800 feet per minute. What angular acceleration must be given to it so that its linear speed may increase to 100 feet per second in 5 seconds ?

12. Find the angular acceleration of a body which increases its angular velocity from 340 revolutions per minute to 500 revolutions per minute in 55 seconds.

13. A point on a revolving body moves at a linear speed which never varies more than 12 per cent. above or below a speed of $\frac{3}{4}$ mile per minute. If the point is 36 inches from the axis of rotation, find the greatest and least angular velocities of the body. If the speed falls from a maximum to a minimum in 25 seconds, what is the angular acceleration ?

14. A body moves from rest along a circular path of one metre radius, with an angular acceleration of ·064 radian per second per second. Calculate its linear velocity and its resultant acceleration after it has been moving for 7 seconds.

15. A railway train moves from rest round a curve of 1,200 feet radius and attains a speed of 15 miles per hour in $2\frac{1}{2}$ minutes. Find (*a*) its linear acceleration, (*b*) its angular acceleration, and (*c*) its normal acceleration at the end of that time.

16. Find the normal acceleration of a point on the rim of a wheel 5 feet 6 inches in diameter revolving at a speed of 600 revolutions per minute.

17. Determine the acceleration of a stone tied to the end of a string 33 inches in length and whirled round at 150 revolutions per minute.

18. A point moves in such a way that the only acceleration it has is one of 2,200 feet per second per second in a direction which is always perpendicular to the direction in which the point is moving. This acceleration increases 20 per cent. when the speed of the point becomes 88 feet per second. What is the radius of the path of the point ? Find also the original linear speed and angular velocity.

19. Find the resultant of an acceleration of 33 feet per second per second and an acceleration of ten miles per minute per minute, the angle between their directions being 60 degrees.

20. A wheel rotates on a horizontal axle with an angular velocity of 900 revolutions per minute. At the same time the frame carrying the axle and wheel rotates about a vertical axis with an angular velocity of 12 revolutions per second. Find the magnitude of the resultant angular velocity of the wheel, and the angle from the horizontal of the resultant axis of rotation.

21. What is the radius of a wheel which rotates at a speed of 216 revolutions per minute, if the normal acceleration of a point on the rim is 2,000 feet per second per second ?

22. A point moves round a circular path of 2 feet 5 inches radius,

and has a normal acceleration of 133 centimetres per second per second. Find (*a*) its angular velocity in revolutions per minute, and (*b*) its linear speed in miles per hour.

23. Find the angular acceleration of a body which increases its angular velocity from 5,370 revolutions per minute to 784 radians per second in 30 seconds.

24. A railway engine travels round a curve of 5 chains radius at a speed of 30 miles per hour. If the driving wheels are 6 feet in diameter, determine the magnitude and direction of their resultant angular velocity. Assume that the track is horizontal.

CHAPTER 8: MOMENTUM AND FORCE

THE quantities with which we have dealt so far have been mostly such as could be explained in terms of our everyday experiences. We now have to consider a quantity which is a little more abstruse, a little more difficult to comprehend, but of very great importance indeed—namely **momentum.**

Momentum is usually defined as the *quantity of motion possessed by a body,* and it is measured by the product of the mass of the body and its linear velocity, i.e.,

$$\text{Momentum} = \text{Mass} \times \text{Velocity}.$$

Since velocity possesses direction as well as magnitude, so momentum, being the product of mass and velocity, possesses both direction and magnitude: it is therefore a *vector quantity.* The direction of the momentum of a body is, of course, the same as the direction of the velocity. We can therefore represent momentum by a straight line or *vector,* and can compound and resolve momenta by means of the **Parallelogram of Momenta.**

MEASUREMENT OF MOMENTUM. Unit momentum is the momentum possessed by unit mass moving with unit velocity. Therefore in the F.P.S. system the unit is the momentum of a mass of one pound, moving with a velocity of one foot per second. In the C.G.S. system the unit is the momentum of a mass of one gramme, moving with a velocity of one centimetre per second. For any body therefore:—

Momentum in F.P.S. units = mass in lbs. × velocity in feet per second.

Momentum in C.G.S. units = mass in gm. × velocity in cm. per second.

There are no special names for the units of momentum.

DIMENSIONS OF MOMENTUM. Since we measure momentum by finding the product of the mass of the body and its linear velocity, the dimensions of momentum will be the product of the dimensions of a mass and a linear velocity, that is, the dimensions will be **[M]** multiplied by $[\mathbf{L}][\mathbf{T}]^{-1}$, that is $[\mathbf{M}][\mathbf{L}][\mathbf{T}]^{-1}$.

CHANGE OF MOMENTUM. Momentum may be considered to be composed of three factors, viz., (i) the *mass* of the body, (ii) the *speed* of the body, and (iii) the *direction* in which it is moving. The momentum will only be constant so long as each of the three factors is constant.

If any one of the factors is changed, therefore, the momentum of the body is necessarily changed. Let us take an example of each kind of change:—

(i) If a loaded motor-lorry is travelling with uniform velocity, and part of its cargo falls off unnoticed, then the momentum of the lorry is reduced because of the reduction in its *mass*, although there is no change in its velocity.

(ii) If the lorry slows down in passing through a town, then its momentum is changed because of the change of *speed*, although there is no change in either the direction of motion or in the mass of the vehicle.

(iii) If the lorry turns a corner, then its momentum is changed because of the change in the *direction* of motion, although its speed remains constant and there is no alteration in its mass.

(iv) We may, of course, have changes of momentum which are due to two or more of the above causes acting at the same time, as for example if the lorry reduces its speed at the same time as it turns a corner.

The important thing to realise is that the momentum must change if any one or more of these factors is changed. The fact that a change of *direction* of motion produces a change of momentum needs special emphasis because it is the factor most likely to escape attention. The most frequently occurring cause of change of momentum is a change in the *speed* of a body, and much of our work

will be concerned with such changes, but the student must never lose sight of the other possible causes of varying momentum.

RATE OF CHANGE OF MOMENTUM. When the momentum of a body is changing, the *rate* of change is the ratio of an indefinitely small change of momentum, to the time occupied in making the change. If, however, the momentum is changing uniformly, then we can obtain the rate of change by simply dividing the total change by the time occupied in making it, i.e.,

$$\text{Rate of change of momentum} = \frac{\text{change of momentum}}{\text{time occupied}}.$$

If the change is not at a uniform rate, then this expression will not give the *actual* but the *average* rate of change of momentum.

LAWS OF MOMENTUM. The principles involved in changes of momentum have for long been known under the name of **Newton's Three Laws of Motion,** because Newton made the first formal statement of them in his "Principia," published in 1686, but they had already been known in various forms before the time of Newton. There are advantages in stating the first two of these laws in a form slightly different from that given by Newton, and terming them the **Laws of Momentum.**[1] Newton's Third Law is really a special case of the second law, as we shall see later.

None of these laws is capable of strict formal proof. They are based on the results of experiment and observation, but they agree so closely with the most accurate work which has been done, that we can accept them without the slightest hesitation.

First Law. *There is no change in the momentum of a body unless the body is acted upon by some external force.*

Second Law. *When there is a change of momentum, then the force producing it is proportional to the rate of*

[1] J. W. Landon, "Elementary Dynamics."

change of momentum and acts in the same direction as the change of momentum.

These laws are of extreme importance in Mechanics, and a great part of our work is based upon them. Let us consider what they really mean.

FIRST LAW OF MOMENTUM. This tells us that a body cannot of itself change its momentum. Stated in that way it appears almost axiomatic, but let us examine it a little closer.

We have already seen that the momentum of a body will change if there is any change in the *mass*, the *speed*, or the *direction of motion* of the body. Therefore, it follows from this law, that none of these factors can change unless some external force acts upon the body, i.e.,

(i) The *mass* can only be changed by some external force.

(ii) The *speed* can only be changed by some external force.

(iii) The *direction of motion* can only be changed by some external force.

With regard to (i) it should be noticed that the mass of a given body is really unalterable. It is a convenience, however, to speak of an alteration in the mass of a body when another body is added to it or part of the body is taken away, as in loading and unloading a vehicle.

From (ii) and (iii) we see that if a body is moving it will continue to move in a straight line for ever, neither slowing down nor turning aside from that straight line, unless some external force compels it to do so. We do not find bodies which actually continue moving uniformly for ever in nature, but when we find a body slowing down, or moving in a curved path instead of a straight one, we can see that it only does so because of the action of some external force, and it is reasonable to conclude that if we could isolate a moving body from any action of external forces, then its motion would continue unchanged for ever, both in magnitude and direction.

The conclusion, then, that we must draw from the First Law of Momentum, is that if we find a body changing its *mass*, or its *speed*, or its *direction of motion*, then we know definitely that it is under the action of some external force. It will be seen from this that if a body is moving in any kind of curved path, then it must be doing so under the compulsion of some definite force: a result that we shall have to consider in more detail later.

It may be noted that the First Law of Momentum is sometimes known as the **Principle of the Conservation of Momentum.**

This law gives us a means of defining **force** in a slightly different way from that which we employed in the second chapter. We can now say that *force is that which produces, or tends to produce a change of momentum in the body upon which it acts.*

SECOND LAW OF MOMENTUM. This tells us that when the momentum of a body changes, the force acting is proportional to the *rate* of change and is in the same direction. As we have already seen, the rate of change of momentum when the change takes place uniformly, or the *average* rate when it is variable, is equal to the total change of momentum divided by the time occupied. Therefore :—

$$\text{Force acting} = \frac{\text{change of momentum}}{\text{time occupied}} \times \text{a constant.}$$

Now if we choose our unit of force suitably, we can make the constant in this equation equal to unity. Then we shall be able to say :—

$$\text{Force} = \frac{\text{change of momentum}}{\text{time occupied}}$$

$$= \frac{\text{mass} \times \text{change of velocity}}{\text{time occupied}}.$$

ABSOLUTE UNITS OF FORCE. We can then say also that unit force is that force which, acting upon

unit mass for unit time, produces unit change of velocity. If we employ absolute units for mass, time, and velocity, then evidently we shall obtain an **absolute unit of force.**

In the F.P.S. system the unit of mass is the *pound,* the unit of velocity is the *foot per second,* and the unit of time is the *second.* Therefore the F.P.S. unit of force must be that force which, acting upon a mass of one pound for one second, gives to it a change of velocity of one foot per second. This force is termed a **poundal.**

In the C.G.S. system the unit of mass is the *gramme,* the unit of velocity is the *centimetre per second,* and the unit of time is the *second.* Therefore the C.G.S. unit of force must be that force which, acting upon a mass of one gramme for one second, gives to it a change of velocity of one centimetre per second. This force is termed a **dyne.**

It will be noticed that, in the above definitions of the poundal and the dyne, it is implied that a change of momentum is due to a change of velocity rather than to a change of mass. As we have already seen, a change of momentum may be due to either cause, but as the great majority of the cases that we shall have to consider will be concerned with changes of velocity without change of mass, it is convenient to concentrate most of our attention on the former.

In all cases, then, where the change of momentum is due to a change of velocity only, we have:—

$$\text{Force} = \frac{\text{change of momentum}}{\text{time occupied}}$$

$$= \frac{\text{mass} \times \text{change of velocity}}{\text{time occupied}}$$

$$= \text{mass} \times \text{acceleration.}$$

This is often a convenient way of summarising the Second Law of Momentum, but it must be remembered that it is not a complete statement of that law. We can use this form of it to modify slightly the definitions

of a poundal and of a dyne, which will then be as follows :—

A **poundal** *is that force which will give an acceleration of one foot per second per second to a mass of one pound.*

A **dyne** *is that force which will give an acceleration of one centimetre per second per second to a mass of one gramme.*

ABSOLUTE AND GRAVITATION UNITS OF FORCE. We are now in a position to compare these absolute units of force with the gravitation units, namely the *pound-weight* and the *gramme-weight.*

We have seen that the acceleration produced by gravity is approximately 32·2 feet per second per second, or 981 centimetres per second per second, and is denoted by the letter **g**. With the knowledge that we have just acquired, this gives us a new means of defining the gravitation units of force, as follows :—

A **pound-weight** *is that force which will give an acceleration of* **g** *feet per second per second to a mass of one pound.*

A **gramme-weight** *is that force which will give an acceleration of* **g** *centimetres per second per second to a mass of one gramme.*

Clearly, then, one pound-weight is equal to g poundals, i.e., to approximately 32·2 poundals, and one gramme-weight is equal to g dynes, i.e., to approximately 981 dynes. We can therefore easily convert forces expressed in gravitation units into their equivalents in absolute units or vice versa :—

$$\left.\begin{array}{ll}\text{One pound-weight} & = g \text{ poundals.} \\ \text{One poundal} & = 1/g \text{ pound-weight.}\end{array}\right\} (g = \text{approx. } 32{\cdot}2).$$

$$\left.\begin{array}{ll}\text{One gramme-weight} & = g \text{ dynes.} \\ \text{One dyne} & = 1/g \text{ gramme-weight.}\end{array}\right\} (g = \text{approx. } 981).$$

CHOICE OF UNITS. When dealing with practical questions—for example, in engineering—it is usually most convenient to employ *gravitation units* of force, but in studying Mechanics it is very much better to use

the *absolute units*. At first this may seem a little difficult, but the advantages are so great that they outweigh any initial trouble involved. Once the habit of employing absolute units has been formed, it will be found of the greatest service in helping us to obtain *clear ideas* and to avoid that confusion of mind which is the bane of the subject of Mechanics.

One of the principal reasons why the absolute unit is preferable to the gravitation unit is that it is *consistent* with the other units of our system of measurement, whether it is the F.P.S. or the C.G.S. system that we are using. This will be realised if we turn back to our definition of the poundal, as that force which will give a velocity of *one* foot per *one* second to a mass of *one* pound when it acts upon it for *one* second. We can only make the *pound-weight* consistent with our other units by altering the whole system and employing a new and highly artificial unit of mass, that is, a unit of mass equal to g pounds. This unit is known as a *slug* and is an objectionable unit, for it depends on the value of g, and therefore *has no constant value*. Any system of units based on a unit of mass which is not a constant quantity stands self-condemned.

Another very important reason for using the absolute unit in preference to the gravitation unit is that it greatly reduces the risk of confusion between **mass** and **weight**, to which we have already drawn attention. If we measure forces in pounds-weight there is always a tendency to abbreviate the name of the unit and to speak of a force of so many *pounds* instead of so many *pounds-weight*. This is very dangerous, because the pound is the unit of *mass*, not of *force*, and therefore its misuse in this way is almost certain to cause confusion of ideas and vagueness of thought. To use the same name for two entirely different quantities is asking for trouble.

We shall, therefore, regularly employ the absolute units of force, the poundal and the dyne, and when there is any occasion to do so, we can, as we have

seen, quickly convert our results into gravitation units.

It will be seen that the weight of a mass of one pound is 32·2 poundals, and the weight of a mass of one gramme is 981 dynes.

The weight of a mass of half-an-ounce is 1/32 of the weight of a mass of one pound, that is, it is a little more than one poundal. We can get an idea of the magnitude of a force of one poundal, therefore, by thinking of the weight of half-an-ounce.

The dyne is an extremely small force, for it takes nearly 14,000 dynes to equal one poundal, and, as we have just seen, one poundal is barely equal to the weight of half-an-ounce.

DIMENSIONS OF FORCE. We can now obtain the dimensions of force, for we see that force is equal to rate of change of momentum, that is, to the change of momentum divided by the time occupied. The dimensions of momentum are $[M][L][T]^{-1}$, and the dimensions of a period of time are $[T]$. Therefore the dimensions of force are $[M][L][T]^{-1}$ divided by $[T]$, that is $[M][L][T]^{-2}$. We can obtain the same result by considering that a force is equal to the product of the mass of a body and the acceleration given to it by the force. The dimensions of a mass are $[M]$, and the dimensions of an acceleration are $[L][T]^{-2}$, so that the dimensions of a force must be $[M]$ multiplied by $[L][T]^{-2}$, that is $[M][L][T]^{-2}$, as before.

The **moment of a force** about a point is equal to the product of the force and its perpendicular distance from that point. Therefore the dimensions of the moment of a force are $[M][L][T]^{-2}$ multiplied by $[L]$, that is $[M][L]^2[T]^{-2}$. The dimensions of **torque** are of course the same.

APPLICATIONS OF THE LAWS OF MOMENTUM. The laws of momentum are of very wide application, and a clear comprehension of them will enable us to deal with many problems in Mechanics with ease and certainty. The essential thing is to make sure that we

really do understand the principles involved, and never under any circumstances to delude ourselves into thinking that a mere memorisation of formulæ is sufficient.

We shall deal with a number of applications of these laws by means of examples, worked out in detail, at the end of this and other chapters. Here we will only consider one case which is typical of many others, and will serve to emphasize the principles.

TWO BODIES CONNECTED BY A STRING. Suppose that we have two bodies fastened to a string which is hung over a pulley, as shown in Fig. 66. For the sake of simplicity we will assume that the string is weightless and inextensible, and that the pulley is fixed and perfectly smooth.

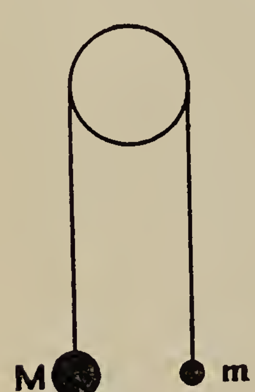


Fig. 66.

Let M pounds be the mass of the heavier body, and m pounds the mass of the other. Then it is clear that M will move downwards and will pull m upwards.

The force of gravity acts on both of the bodies. It acts on the mass M with a force equal to Mg poundals, and on the other mass with a force of mg poundals, Mg and mg being the respective weights of the masses. These two forces tend to move the string in opposite directions, and their resultant will therefore be $(Mg - mg)$, that is $(M - m)g$ poundals, in the same direction as the greater force Mg.

The net effective *force* acting upon the system is therefore $(M - m)g$ poundals, and the *mass* upon which it acts is $(M + m)$ pounds, since both masses must move at the same time and to the same extent, and therefore with the same acceleration.

By the Second Law of Momentum :—

$$\text{Force} = \text{mass} \times \text{acceleration},$$

$$\text{whence Acceleration} = \frac{\text{force producing motion}}{\text{mass on which it acts}}.$$

Therefore in this case we shall have :—

$$\text{Acceleration} = \frac{(\text{M} - m)g \text{ poundals}}{(\text{M} + m) \text{ pounds}}$$

$$= \frac{\text{M} - m}{\text{M} + m}.g \text{ feet per sec. per sec.}$$

We can look at the matter in a slightly different way. Consider the forces acting upon the mass M. They are, its weight Mg poundals acting vertically downwards, and the tension, T, of the string acting vertically upwards. The *difference* between these two forces is the effective force producing the motion of the mass M. Therefore we have :—

Acceleration of larger mass M downwards

$$= \frac{\text{force producing motion}}{\text{mass on which it acts}}$$

$$= \frac{\text{M}g - \text{T}}{\text{M}} \text{ feet per second per second.}$$

Now consider the forces acting upon the other mass, m. They are, its weight mg poundals acting vertically downwards, and the tension of the string, T poundals, acting vertically upwards. The effective force producing motion is the *difference* between these two forces, and will evidently be a force of $(\text{T} - mg)$ poundals acting upwards. Therefore :—

Acceleration of smaller mass m upwards

$$= \frac{\text{force producing motion}}{\text{mass on which it acts}}$$

$$= \frac{\text{T} - mg}{m} \text{ feet per second per second.}$$

Clearly, since the two masses are connected by the string, the acceleration of the larger mass M downwards must be equal to the acceleration of the smaller mass upwards, i.e.,

$$\frac{\text{T} - mg}{m} = \frac{\text{M}g - \text{T}}{\text{M}},$$

whence $M(T - mg) = m(Mg - T)$

$$MT - Mmg = Mmg - mT$$

$$MT + mT = 2Mmg$$

and $$T = \frac{2Mm}{M + m}.g \text{ poundals,}$$

from which result we can, by substitution, obtain the value of the acceleration as before.

This example is worth studying carefully as it illustrates the Second Law of Momentum very clearly. In any such case we must determine what are the *forces* acting upon each body, and what is its *mass*, taking particular care that we do not confuse the two quantities. We ought always to get our information accurately and completely stated, *and checked,* before we commence to manipulate it.

IMPULSE. The **impulse** of a force may be defined as the *total change of momentum which it produces.* Therefore :—

Impulse = force × time during which it acts.

It should be noted that if the force is a variable one, we must take its mean value.

We have :—

Impulse = force × time
= mass × acceleration × time
= mass × velocity imparted
= momentum imparted.

We give this special name of *impulse* to the change of momentum produced by a force, for convenience in dealing with cases where the change of momentum takes place suddenly. If we have a very great force acting for only a very short time, then it may be impossible, or at least very difficult, to measure either the force itself or the time during which it acts. Any such force we term an **impulsive force,** and we measure its effect simply by the change of momentum it produces, that is, its impulse.

The force exerted by the head of a hammer in striking a nail is an example of an impulsive force. The hammer-head descends with considerable velocity, and therefore with considerable momentum, and this momentum being destroyed in an extremely short period of time, the force exerted upon the nail is a very great one. It would be very difficult to measure the force or the time in such a case, but the impulse could be determined with relative ease.

DIMENSIONS OF IMPULSE. We may define impulse shortly as the product of a force and the time during which it acts. The dimensions of a force are $[M][L][T]^{-2}$, and those of a period of time are $[T]$. Therefore the dimensions of an impulse will be $[M][L][T]^{-2}$ multiplied by $[T]$, that is $[M][L][T]^{-1}$. These are the same dimensions as we have already obtained for *momentum*, which is what we should expect since we may define impulse as change of momentum.

IMPACT OF TWO BODIES. When two bodies collide, the force exerted by either body upon the other is an impulsive one, and therefore we measure its effect by determining the change of momentum it produces. If we consider the two bodies together as forming a single system, then, since there are no *external* forces acting upon the system, the *sum* of the momenta of the two bodies must remain constant, that is, the momentum of the *system* must remain constant. This follows necessarily from the First Law of Momentum (or Principle of the Conservation of Momentum).

Therefore we have :—

Total momentum of system after impact
= total momentum of system before impact :

also Gain of momentum by one body
= loss of momentum by the other body.

This is a very useful result and it should be carefully noted.

NEWTON'S THIRD LAW OF MOTION. By the Second Law of Momentum, the force acting upon a body is equal to its rate of change of momentum. When two bodies act upon each other, then, as shown above, the gain of momentum by one of the bodies is equal to the loss of momentum by the other, and clearly the two changes of momentum must take place in the same time.

It follows that the rate of change of momentum of one of the bodies must be equal in magnitude and opposite in direction to the rate of change of momentum of the other body: i.e., the forces must be equal and opposite. This result is usually known as **Newton's Third Law of Motion,** and is then stated in the form :—

Action and reaction are equal and opposite.

As we have seen, it is really only a corollary to the First and Second Laws of Momentum. Its application is not limited to the forces exerted between colliding bodies, nor is it even limited to forces which produce motion: it can be applied equally to forces which do not produce motion. In such cases the rate of change of momentum of the system as a whole, and of each part of it separately, must be zero, and therefore every force acting upon the system must be balanced by an equal and opposite reaction.

SUMMARY OF CHAPTER 8

The **momentum** of a body is the quantity of motion which it possesses: it is equal to the product of the *mass* of the body and the *velocity* of the body.

Momentum is a vector quantity and therefore momenta can be compounded and resolved by means of the **Parallelogram of Momenta.**

Momentum in F.P.S. units
= mass in pounds × velocity in feet per second.

Momentum in C.G.S. units
= mass in grammes × velocity in centimetres per second.

The dimensions of momentum are $[M][L][T]^{-1}$.

Any change in the *mass*, or the *speed*, or the *direction of motion* of a body, produces a change in its momentum.

The (uniform or mean) **rate of change of momentum** of a body is obtained by dividing the change of momentum by the time occupied in making it.

First Law of Momentum. There is no change in the momentum of a body unless the body is acted upon by some external force.

Second Law of Momentum. When there is a change of momentum, then the force producing it is proportional to the rate of change of momentum and acts in the same direction as the change of momentum.

If a body moves in any path which is not a straight line, then it must be doing so under the compulsion of some definite force.

In cases where the change of momentum of a body is due to a change of velocity, we may state that :—

$$\text{Force} = \text{mass} \times \text{acceleration}.$$

A **poundal** is that force which will give an acceleration of one foot per second per second to a mass of one pound.

A **dyne** is that force which will give an acceleration of one centimetre per second per second to a mass of one gramme.

A **pound-weight** is that force which will give an acceleration of **g** feet per second per second to a mass of one pound.

A **gramme-weight** is that force which will give an acceleration of **g** centimetres per second per second to a mass of one gramme.

One pound-weight is equal to **g** poundals (g = approx. 32·2).

One gramme-weight is equal to **g** dynes (g = approx. 981).

The *poundal* and the *dyne* are **systematic absolute units.** The *pound-weight* and the *gramme-weight* are **variable units.**

The dimensions of a force are $[M][L][T]^{-2}$. The dimensions of a torque or turning-moment are $[M][L]^2[T]^{-2}$.

The **impulse** of a force is the total change of momentum which it produces. An **impulsive force** is one which produces a sudden change of momentum. The dimensions of impulse are the same as those of momentum.

When two bodies collide, the *total* momentum of the system is unchanged by the impact. Therefore the gain of momentum by one body is equal to the loss of momentum by the other.

Action and reaction are equal and opposite (Newton's Third Law of Motion).

EXAMPLES VIII

(*For Hints on Working Examples, see page* 21.)

1. *Find the momentum, in F.P.S. units, of a motor-car of* 25 *cwt. mass, moving with uniform velocity at* 18 *miles per hour.*

Momentum = mass × velocity

$$= (25 \times 112) \text{ pounds} \times \left(18 \times \frac{22}{15}\right) \text{ feet per second per second}$$

$$= \frac{25 \times 112 \times 18 \times 22}{15}$$

$$= 7{,}392 \text{ F.P.S. units.}$$

2. *Determine the force necessary to give an acceleration of* $1\frac{1}{4}$ *miles per minute per minute to a mass of* ·09 *ton.*

Force required

$$= \text{mass} \times \text{acceleration}$$

$$= \cdot 09 \text{ ton} \times 1\tfrac{1}{4} \text{ miles per minute per minute}$$

$$= (\cdot 09 \times 2{,}240) \text{ pounds} \times \left(\frac{1\frac{1}{4} \times 5{,}280}{60 \times 60}\right) \text{ feet per second per second}$$

$$= 201\cdot 6 \text{ pounds} \times \frac{11}{6} \text{ feet per second per second}$$

$$= 369\cdot 6 \text{ poundals.}$$

3. *A force of one million dynes acts on a mass of* 850 *kilogrammes. Find how long it will take for the body to attain a velocity of* 96 *kilometres per hour. What will then be the momentum of the body?*

Acceleration imparted to body

$$= \frac{\text{force acting upon body}}{\text{mass of body}}$$

$$= \frac{1{,}000{,}000 \text{ dynes}}{850{,}000 \text{ grammes}}$$

$$= \frac{20}{17} \text{ centimetres per second per second.}$$

Velocity attained

$$= 96 \text{ kilometres per hour}$$

$$= \frac{8{,}000}{3} \text{ centimetres per second.}$$

Time required to attain given velocity

$$= \frac{\text{velocity}}{\text{acceleration}}$$

$$= \left(\frac{8{,}000}{3} \times \frac{17}{20}\right) \text{ seconds}$$

$$= \frac{6{,}800}{3} \text{ seconds}$$

$$= \frac{6{,}800}{3 \times 60} \text{ minutes}$$

$$= 37{\cdot}8 \text{ minutes.}$$

Momentum acquired

$$= \text{mass} \times \text{velocity}$$

$$= 850{,}000 \text{ grammes} \times \frac{8{,}000}{3} \text{ centimetres per second}$$

$$= 2{,}267 \text{ million C.G.S. units.}$$

4. *An inextensible weightless string passing over a smooth pulley carries at each end a scale-pan of ½ pound mass. In one pan is a disc of brass of 2 pounds mass, and in the other a disc of copper of 3 pounds mass. Find the tension in the string, the acceleration with which the system will move, and the pressure between each scale-pan and the disc it carries.*

Let T = tension in string in poundals.

Consider the motion of the scale-pan with the brass disc. We have :—

$$\text{Acceleration} = \frac{\text{force causing motion}}{\text{mass of body moved}}$$

$$= \frac{\text{tension in string} - \text{weight of disc and pan}}{\text{mass of disc and pan}}$$

$$= \frac{(T - 2\tfrac{1}{2}g) \text{ poundals}}{2\tfrac{1}{2} \text{ pounds}}$$

$$= \frac{T - 80{\cdot}5}{2{\cdot}5} \text{ feet per second per second.}$$

In the case of the scale-pan with the copper disc, we have :—

$$\text{Acceleration} = \frac{\text{force causing motion}}{\text{mass of body moved}}$$

$$= \frac{(3\tfrac{1}{2}g - T) \text{ poundals}}{3\tfrac{1}{2} \text{ pounds}}$$

$$= \frac{112{\cdot}7 - T}{3{\cdot}5} \text{ feet per second per second.}$$

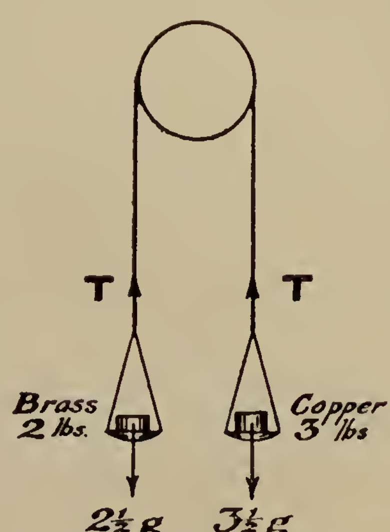


FIG. 67.

The accelerations in each of these cases must be equal since the string is inextensible. Therefore :—

$$\frac{T - 80{\cdot}5}{2{\cdot}5} = \frac{112{\cdot}7 - T}{3{\cdot}5},$$

whence
$$3{\cdot}5(T - 80{\cdot}5) = 2{\cdot}5(112{\cdot}7 - T)$$
$$3{\cdot}5T - 281{\cdot}8 = 281{\cdot}8 - 2{\cdot}5.T$$
$$6.T = 563{\cdot}6$$
and
$$T = 93{\cdot}93 \text{ poundals,}$$

i.e., tension in string $= 93{\cdot}93$ poundals.

$$\therefore \text{ Acceleration} = \frac{T - 80{\cdot}5}{2{\cdot}5}$$
$$= \frac{93{\cdot}93 - 80{\cdot}5}{2{\cdot}5}$$
$$= 5{\cdot}37 \text{ feet per second per second.}$$

This result might also be derived directly from the equation of motion for the whole system :—

$$\text{Acceleration} = \frac{\text{force producing motion}}{\text{mass of bodies moved}}$$
$$= \frac{(3\tfrac{1}{2}g - 2\tfrac{1}{2}g) \text{ poundals}}{(3\tfrac{1}{2} + 2\tfrac{1}{2}) \text{ pounds}}$$
$$= \frac{g}{6}$$
$$= 5{\cdot}37 \text{ feet per second per second, as before.}$$

Upward force on brass disc to balance its weight
$$= 2 \times 32{\cdot}2$$
$$= 64{\cdot}4 \text{ poundals.}$$

Upward force on brass disc to produce motion
$$= \text{mass of disc} \times \text{acceleration}$$
$$= 2 \text{ pounds} \times 5{\cdot}37 \text{ feet per second per second}$$
$$= 10{\cdot}74 \text{ poundals.}$$

$\therefore$ Total upward force on brass disc
$$= 64{\cdot}4 \text{ poundals} + 10{\cdot}74 \text{ poundals}$$
$$= 75{\cdot}14 \text{ poundals.}$$

This is therefore the pressure between the scale-pan and the brass disc.

Similarly, the pressure between the scale-pan and the copper disc :—
$= $ force to balance weight of disc — force to produce motion
$= (3 \times 32{\cdot}2)$ poundals $-$ $(3 \times 5{\cdot}37)$ poundals
$= 96{\cdot}6 - 16{\cdot}1$
$= 80{\cdot}5$ poundals.

The same results can be obtained in a slightly different manner. The tension in the string has to balance the weight of the brass disc and the scale-pan supporting it. It has also to give acceleration to both of these. Clearly, the disc and the pan will share the tension of the string in proportion to their respective masses, so that the upward pressure on the disc, that is, the pressure between the scale-pan and disc :—

$$= \frac{\text{mass of disc}}{\text{mass of disc and pan}} \times \text{tension in string}$$

$$= \frac{2 \text{ pounds}}{2\frac{1}{2} \text{ pounds}} \times 93{\cdot}93 \text{ poundals}$$

$$= 75{\cdot}14 \text{ poundals, as before.}$$

Similarly, pressure between scale-pan and copper disc :—

$$= \frac{3 \text{ pounds}}{3\frac{1}{2} \text{ pounds}} \times 93{\cdot}93 \text{ poundals}$$

$$= 80{\cdot}5 \text{ poundals, as before.}$$

5. Find the momentum of a body of 450 pounds mass moving with a velocity of 63 feet per second.

6. Determine the momentum of a body of 54 kilogrammes mass moving with a velocity of 1,400 centimetres per second.

7. What is the momentum of a locomotive weighing $82\frac{1}{2}$ tons travelling due North at 50 miles per hour ?

8. Find how long it will take to destroy the momentum of a body of $\frac{3}{4}$ ton mass, moving with a velocity of 5 furlongs a minute, by means of a force of 560 pounds-weight.

9. Determine the force necessary to give an acceleration of $\frac{1}{4}$ foot per second per second to a mass of one ton.

10. A force of 1,000 poundals acts upon a mass of 14·7 tons. Find the velccity and momentum of the body at the end of half-an-hour.

11. Calculate the force required to impart a velocity of 60 miles per hour to a train of 240 tons mass in 5 minutes.

12. What must be the retarding force per ton mass exerted by the brakes of a train in order that when travelling at a speed of 52 miles per hour it may be brought to rest in $3\frac{1}{2}$ minutes ? How far will the train travel in that time ?

13. A train of 330 tons mass is travelling at a speed of 70 miles per hour when it is required to slow down to a speed of 15 miles per hour. Calculate the retarding force necessary in order that the

reduction of speed may be made within a distance of two miles after the brakes are applied.

14. An inextensible weightless string passing over a smooth pulley carries at each end a hook weighing 6 ounces. On one of these hooks is hung a lead ball weighing 4 pounds, and on the other a hammer-head of unknown weight. It is found that the lead ball rises with an acceleration of 5 feet per second per second. Find the mass of the hammer-head, the tension in the string, and the pressure between each hook and the load which it carries.

15. A force of 800 poundals acts upon a mass of $2\frac{1}{4}$ tons for a period of one minute. A force of 30 pounds-weight then acts upon the body in the opposite direction from that of the first force for a period of 50 seconds. Find the magnitude and direction of the velocity of the body at the end of the second period. What is then its momentum ?

16. What acceleration will be given to a train of 275 tons mass by a locomotive which can exert for that purpose a draw-bar pull of 9,000 pounds-weight ? How long will it take to haul the train $1\frac{1}{2}$ miles from rest ?

17. A force of 70 poundals acts on a body A, of 32 pounds mass, for 20 seconds. A force of $12\frac{1}{2}$ pounds-weight acts on a body B, of 68 pounds-mass, for half-a-minute. Find (i) the ratio of the velocity imparted to B to the velocity imparted to A, and (ii) the ratio of the momentum imparted to B to the momentum imparted to A.

18. A train is crossing a viaduct, with constant velocity of 37·5 miles per hour, when a ball weighing $5\frac{1}{2}$ ounces is dropped from the window of one of the coaches. If the ball falls clear of the viaduct, find the magnitude and direction of its momentum (*a*) one second later, and (*b*) two seconds later.

19. Given that one foot equals 30·48 centimetres, and one ounce equals 28·35 grammes, find the ratio of the F.P.S. unit of momentum to the C.G.S. unit of momentum.

20. A motor-car of 2,400 pounds mass is travelling Northwards with a constant speed of 22·5 miles per hour. A little later it is travelling towards the North-East at a speed of 25·8 miles per hour. Find the change of momentum it has received.

21. How fast must a truck of $\frac{1}{2}$ cwt. mass be moving so that it may have momentum of the same magnitude as a bullet which is travelling at 1,400 feet per second, and of which the mass is one ounce ?

22. Two balls, each of one pound mass, collide when travelling

in opposite directions at a speed of 5 miles per hour. If the collision occupies 1/2000 second, find the force exerted between the balls.

23. Find the force required to give a velocity of 43 miles per hour to a train of 220 tons mass in $1\frac{3}{4}$ minutes.

24. What is the acceleration of a mass of 17 tons under the action of a force of 37 pounds-weight ? How long will it take for the mass to acquire a momentum of three million F.P.S. units, and what will then be its velocity ?

CHAPTER 9: FRICTION

IF we try to make one body slide over another we always encounter a certain amount of resistance. If the body is a heavy one, the resistance may be sufficiently great to prevent us from moving the body at all. For example, if we try to move a piece of furniture across a room, we find that considerable effort is required, especially if the floor of the room is not very smooth and the article is not fitted with castors. Again, if we try to turn a key in a lock which has been out of use for some time, we find that it turns with great difficulty and much rasping, and we say that the lock needs oiling.

This natural resistance to the sliding of one body over another is termed **friction**, and is, of course, a *force*.

The exact nature of friction, from a scientific point of view, has been much discussed, but is not yet fully understood. Various theories have been suggested to account for the phenomena, but none of them are entirely satisfactory. In studying Mechanics, however, we are more concerned with the practical aspects of friction than with the theoretical; that is, we have to consider the phenomena themselves and the laws which they follow, rather than the underlying causes.

Everybody knows that the resistance to sliding is less with some kinds of material than with others: the boy who makes a "slide" on a frosty day, is well aware of this; he knows that his feet will slide on ice in a way that they would never do on paving-stones for instance. Similarly, one does not carpet the floor of a ball-room: the feet of the dancers glide much more easily over polished wood. The amount of friction depends then on the *material* of the surfaces which slide over one another.

Even, however, with the same kinds of material, the amount of friction is not the same under all conditions.

The schoolboy finds his slide improved when repeated use has rendered the surface of the ice smoother. The dancer likes the floor to be kept well polished, and free from any roughness of surface. We see, therefore, that the amount of friction, that is, the magnitude of the force of friction, depends also on the *condition* of the surfaces, as well as the material of which they are composed; that is, it depends upon the smoothness or roughness of the surfaces.

Another characteristic of friction that is familiar to us all, is that the resistance is greater when we slide a heavy body over a surface than when we slide a light body over the same surface. One could easily pull an empty box along on even a rough surface, but with a heavy load inside the same box, it might be almost impossible to move it at all. The amount of friction is therefore dependent on the force with which the two surfaces are pressed together, that is, on the *pressure* between the surfaces.

Yet another characteristic of friction that we very soon notice is that it *always opposes motion*. In whatever direction we try to make a body slide, friction will always act in the opposite direction, and retard or prevent motion. In this it reminds us of the people one sometimes meets who oppose any and every suggestion made to them, from sheer contrariness!

LIMITING FRICTION. When we investigate experimentally the properties and effects of friction, we find that, in any given case, some definite force is required to overcome the resistance of friction and to produce motion. If we apply any less force, the resistance balances the force applied, and the body remains at rest. If we do not apply any force at all, the body remains equally at rest.

Clearly, then, friction is a force of that kind which we termed **passive forces**, because they can never originate motion. As we have already seen, passive forces are merely called into play by those of the active kind:

they have no independent existence. If there is no force attempting to move one body over the surface of another, then there is no friction between the two bodies. As soon, however, as we try to make one of the bodies slide over the other, friction is called into action and opposes our efforts up to a certain limit, beyond which it can do no more. When that limit is passed, equilibrium breaks down and motion takes place. Friction has not ceased to act, but it has reached its maximum value and has been overcome by a greater force. This maximum value we term **limiting friction.**

LAWS OF FRICTION. We see that the phenomena of friction are governed by definite rules or laws. These phenomena were investigated nearly a century ago by Morin and Coulomb, and as a result of their extensive experiments a number of conclusions were reached, which have been known for many years as the **Laws of Friction.**

The experiments of Morin and Coulomb, although made with great care, were restricted in scope, and the laws based on their results have since been shown to be true only within certain limitations. The conventional form in which these laws are usually stated, though convenient in some respects, tends to obscure certain important considerations and to ignore others : we shall therefore state them in a somewhat different way. We shall only deal here with surfaces which are **plane** and **dry.**

Friction is a force : it must therefore necessarily possess the three properties of **magnitude, direction,** and **sense.** Let us, then, state the laws of friction as they affect these three properties.

The **magnitude** of the force of friction between two plane, dry surfaces is equal to the magnitude of the force which tends to produce sliding between the surfaces, up to a certain value which is known as the **limiting friction.** This maximum value :—

(*a*) depends on the nature of the surfaces ;

(*b*) depends on the condition of the surfaces;
(*c*) is directly proportional to the normal pressure between the surfaces, unless the pressure is very great;
(*d*) is nearly independent of the speed of sliding when sliding takes place at low speeds, but is somewhat greater just before motion commences than during motion;
(*e*) is approximately independent of the area of the surfaces in contact.

The **direction** of the force of friction between two surfaces :—

(*a*) is in the plane of the surfaces;
(*b*) is parallel to the direction of motion or the direction in which it is attempted to produce motion.

The **sense** of the force of friction between two surfaces is always contrary to that of the force which produces, or attempts to produce, sliding.

It will be seen that there are quite a number of factors which have to be taken into account in considering the behaviour of the force of friction. The most important of these may be brought together and summarised as follows *for a given pair of surfaces* and for *limiting* friction :—

(*a*) The *magnitude* is directly proportional to the normal pressure between the surfaces.

(*b*) The *direction* is precisely opposite to the direction of the actual or attempted motion of the sliding body.

COEFFICIENT OF FRICTION. We see, then, that the ratio of the force of friction to the normal pressure between the surfaces is constant for a given pair of surfaces, i.e., that :—

$$\frac{\text{Force of friction}}{\text{Normal pressure}} = \text{a constant.}$$

To this constant we give the name **coefficient of**

friction, and we usually denote it by the Greek letter μ (pronounced "mu").

It must be carefully noticed that the coefficient of friction is calculated from the *limiting* or maximum value of the force of friction, that is, the value of the force of friction when motion is taking place or is on the point of taking place. Obviously, friction may have any *less* value than this maximum.

We may say, then, that the maximum force of friction is equal to the coefficient of friction multiplied by the normal pressure between the surfaces.

It should be noted that μ is always less than unity, and therefore the maximum force of friction is always less than the normal pressure between the surfaces.

EXPERIMENTAL DETERMINATION OF COEFFICIENT OF FRICTION. We can determine experimentally the value of the coefficient of friction for a particular pair of surfaces, by means of the apparatus shown in Fig. 68. A is a board or slab of one of the materials to be tested, and is fixed in a horizontal position on a bench or table. B is a slider or tray composed of the other material to be tested, and is placed on the board A. To this slider B is attached a thin cord C, which passes over a pulley D and has a scale-pan attached in which weights can be placed.

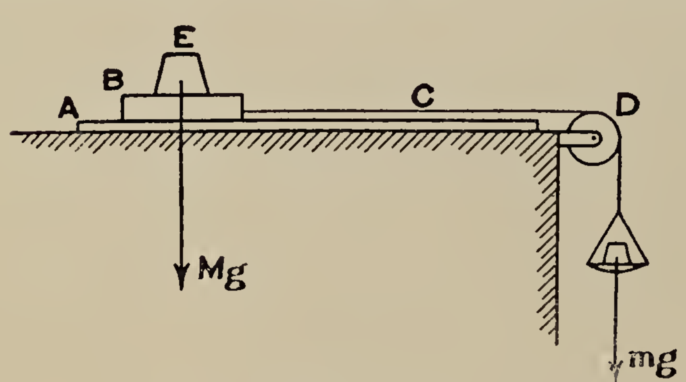


Fig. 68.

Sufficient weights are placed in the scale-pan so that the slider, when started, just continues to move uniformly along the board A. Then, if M is the mass of the slider and its load E, the normal pressure between the surfaces of A and B is the weight Mg. Also, if m is the mass of the scale-pan and its contents, the tension in the string is equal to the weight mg. Clearly, the weight mg measures the force which is required to overcome the force of

friction when the normal pressure between the surfaces is Mg (Fig. 69).

The experiment is repeated with various loads on the slider in turn, the corresponding value of the weights in the pan being noted in each case. It will be found that, within the limits of experimental error, the value of the ratio *mg*/Mg, that is, the ratio *m*/M, is constant for the same pair of surfaces. This value is of course μ, the coefficient of friction for the pair of surfaces.

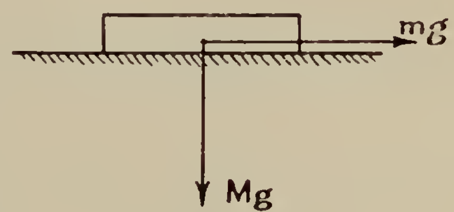


FIG. 69.

On repeating the experiment with a different pair of surfaces a different value of μ will be obtained, depending upon the nature and condition of the materials of which they are composed.

ALTERNATIVE EXPERIMENT. There is another simple way in which we can obtain experimentally the value of μ, the coefficient of friction, for a pair of plane surfaces.

A board carrying one of the surfaces is so arranged that it can be tilted to any required angle of slope, the angle being indicated by a pointer moving over a scale (see Fig. 70). A slider, which forms the other surface, is placed on the plane, which is then tilted until its inclination to the horizontal is such that the slider, if just given a start, will continue to move down the plane at uniform speed. The angle of inclination ϕ (phi) of the plane is then noted. The experiment is then repeated with various loads placed on the slider, and the angle ϕ noted in each case.

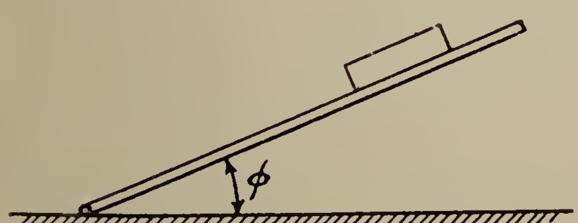


FIG. 70.

Let us consider the forces acting upon the slider. First there is its **weight** (including the weight of any load placed upon it) Mg, acting vertically downwards, as shown in Fig. 71. Then there is the force of **friction,**

F, opposing motion, as always, and therefore acting up the plane. Lastly there is the normal **reaction,** N, acting perpendicularly from the plane on to the slider.

These three forces must be in equilibrium (for the motion is uniform), and therefore, since they are not all parallel, they must all meet at some one point. Now if we have three forces in equilibrium acting at a point, then they can be represented by the sides of a triangle taken in order (the *Triangle of Forces*).

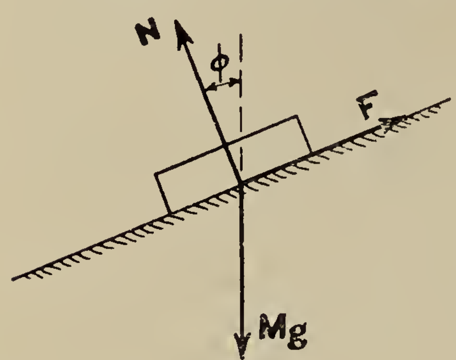


Fig. 71.

Drawing AB, then, to represent the weight, Mg, both in magnitude and direction, as shown in Fig. 72, we complete the triangle of forces by drawing BC normal to the direction of slope of the plane, to represent the direction of the normal reaction, N, and AC parallel to the direction of the slope to represent the direction of the force of friction, F. These two lines will intersect at C, and then BC represents the normal reaction, N, both in magnitude and direction, and CA represents the force of friction, F, both in magnitude and direction.

Fig. 72.

Clearly the angle ABC is equal to the angle of slope, ϕ, of the plane, since AB is vertical and BC is normal to the plane. Then we have:—

Coefficient of friction

$$= \mu = \frac{\text{Force of Friction}}{\text{Normal pressure}}$$

$$= \frac{F}{N}$$

$$= \frac{CA}{BC}$$

$$= \tan \phi$$

The angle ϕ is termed the **angle of friction**, and we have therefore the important result :—

The coefficient of friction is equal to the tangent of the angle of friction.

We can confirm this experimentally by using the same pair of surfaces, and the same loading, for both of the above experiments. It will be found that, within the limits of experimental error :—

$$\underset{\text{(from first experiment)}}{\mu} = \underset{\text{(from second experiment).}}{\tan \phi}$$

FRICTION IN MACHINES. In machinery of all kinds we always have friction between the various working parts : sometimes much and sometimes little, according to circumstances, but always present. In order to overcome this friction we have to do more work than would otherwise be necessary, and therefore, in most cases, we strive to reduce the amount of friction as much as possible. Generally, the less friction we have in our machines, the more efficiently and economically they can be worked.

Friction is, however, by no means always a disadvantage. If all friction could be abolished we should have to revise our ideas considerably and alter our whole method of living. For example, if it were not for friction between our feet and the ground we could not walk. If there were no friction between the wheels and the rails, a railway locomotive would be unable to start moving, nor, once started, could it easily be stopped, since the brakes would be useless.

If there were no such thing as friction, nails and screws would be useless, or nearly so : nor would our ordinary method of driving machinery by means of belting serve us any longer, for the belts would slip uselessly round the pulleys without moving them at all. Evidently, therefore, friction is only objectionable when we have it in the wrong places.

We shall return to the subject of friction in machines later, when considering work and energy.

TRACTIVE RESISTANCE. When a vehicle is in motion with uniform velocity, no force is required to *produce* motion for there is no acceleration of any kind. Yet, as we know, force is actually needed to *maintain* the motion. This force, then, merely overcomes the resistances that would otherwise bring the vehicle to rest.

These resistances are due to a variety of causes. There is **friction** between the wheels and their bearings, and between other parts of the vehicle: there is **rolling resistance** between the wheels and the road or rails upon which they run; and there is **air resistance** also.

It would be a difficult matter in most cases to determine the exact amount of each of these resistances separately, and there would not be much advantage, for ordinary purposes, in doing so. We therefore include them all in the one term **tractive resistance,** without distinguishing between them.

It will easily be seen that it is a relatively simple matter to determine the amount of this tractive resistance in any particular case, for we simply have to find what force will suffice to maintain motion of the vehicle at any given velocity. The results of experiments of this kind are usually expressed by stating the force in pounds-weight necessary to maintain the motion of each ton of the mass of the vehicle: that is, we say that the tractive resistance is so many pounds-weight per ton mass of vehicle.

The amount of tractive resistance depends considerably upon the speed of the moving vehicle. For railway trains it has been found that the resistance is fairly high when motion is only just commencing, and that it decreases as the speed increases up to about 5 miles per hour, when it is a minimum. Beyond this speed it increases again and continues to do so for all higher speeds.

Tractive resistance resembles friction in many respects: it is a passive force: it opposes motion: its magnitude depends approximately upon the weight

of the vehicle. For many purposes, therefore, we can treat it as though it were simply friction.

SUMMARY OF CHAPTER 9

Friction is the resistance offered between the surfaces of two bodies when it is attempted to make one of them slide over the other. It is a passive force, being only called into play by a force of the active kind.

The magnitude of the force of friction is exactly equal to that of the force which calls it into play, up to a certain maximum value, which is termed **limiting friction.**

This limiting value depends upon the nature and condition of the surfaces in contact, and is directly proportional to the normal pressure between those surfaces.

The direction of the force of friction is always exactly opposite to that of the force which produces, or attempts to produce, sliding between the surfaces.

For a given pair of surfaces, the ratio of the *limiting* force of friction to the normal pressure between the surfaces, is constant. This constant ratio is termed the **coefficient of friction** for the pair of surfaces, and is denoted by the Greek letter μ. μ is always less than unity.

The greatest angle of inclination (from the horizontal), at which one plane surface will rest on another without sliding, is known as the **angle of friction** and is denoted by the Greek letter ϕ. For any pair of surfaces :—

$$\mu = \tan \phi$$

Friction in machinery is usually a disadvantage, and we therefore attempt to reduce it as much as possible in the great majority of cases : there are, however, many instances in which friction serves useful purposes.

Tractive resistance is the term which we employ to include all the resistances to the motion of a vehicle, apart from the resistances to starting or accelerating. It includes *friction, rolling resistance,* and *air resistance,* and is usually measured in pounds-weight per ton mass of vehicle. The amount of the tractive resistance in any given case depends upon the speed, other things being equal.

EXAMPLES IX

(*For Hints on Working Examples, see page* 21.)

1. *A block of wood of* 3·72 *pounds mass rests on a horizontal board. If the horizontal force required to make the block slide is* 24·7 *poundals, determine the coefficient of friction for the pair of surfaces.*

If a 7-pound iron weight is placed on top of the block, what force will now be required to produce sliding?

Coefficient of friction

$$= \frac{\text{force of friction}}{\text{normal pressure between surfaces}}$$

$$= \frac{24 \cdot 7 \text{ poundals}}{(3 \cdot 72 \times 32 \cdot 2) \text{ poundals}}$$

$$= \frac{24 \cdot 7}{119 \cdot 9}$$

$$= \cdot 206.$$

Force required to move loaded block

$$= \text{weight} \times \text{coefficient of friction}$$

$$= (10 \cdot 72 \times 32 \cdot 2) \text{ poundals} \times \cdot 206$$

$$= 71 \cdot 2 \text{ poundals.}$$

2. *A loaded brass slider requires a force of* 1·98 *pounds-weight to move it against the resistance of friction on a horizontal plane. The slider weighs* 1·54 *pounds and its load weighs* 9 *pounds. What is the coefficient of friction? If the plane were gradually tilted, at what angle would the slider begin to move down the plane?*

Coefficient of friction

$$= \frac{\text{force of friction}}{\text{normal pressure between surfaces}}$$

$$= \frac{1 \cdot 98 \text{ pounds-weight}}{(1 \cdot 54 + 9) \text{ pounds-weight}}$$

$$= \cdot 188.$$

Tangent of angle of friction

$$= \text{coefficient of friction}$$

$$= \cdot 188$$

$$= \tan 10^\circ 40'.$$

The slider would therefore begin to move down the plane when the latter was inclined at $10^\circ 40'$ to the horizontal.

3. *What draw-bar pull must a railway locomotive exert in order to attain a speed of* 45 *miles per hour in* 3 *minutes from rest, when hauling a train of* 300 *tons mass against a mean tractive resistance of* 12 *pounds-weight per ton?*

Force required to overcome friction, etc.

$$= 12 \text{ pounds-weight per ton} \times 300 \text{ tons}$$

$$= 3{,}600 \text{ pounds-weight.}$$

Force required to produce acceleration

$$= \text{mass} \times \text{acceleration}$$

$$= 300 \text{ tons} \times \frac{45 \text{ miles per hour}}{3 \text{ minutes}}$$

$$= (300 \times 2{,}240) \text{ pounds} \times \frac{66 \text{ feet per second}}{180 \text{ seconds}}$$

$$= \frac{300 \times 2{,}240 \times 66}{180 \times 32{\cdot}2} \text{ pounds-weight}$$

$$= 7{,}650 \text{ pounds-weight.}$$

∴ Draw-bar pull required

$$= 3{,}600 \text{ pounds-weight} + 7{,}650 \text{ pounds-weight}$$

$$= 11{,}250 \text{ pounds-weight}$$

$$= 5{\cdot}02 \text{ tons-weight.}$$

4. *A block of wood rests on a board, and it is found that when the board is tilted to an angle of* 20 *degrees to the horizontal, the block is on the point of slipping. How long will it take for the block to slide* 8 *feet down the board when it is tilted to an angle of* 30 *degrees?*

Coefficient of friction

$$= \mu = \text{tangent of angle of friction}$$

$$= \tan 20^\circ$$

$$= {\cdot}364.$$

Force of friction

$$= \text{coefficient of friction} \times \text{normal reaction of plane}$$

$$= \mu \times \text{Mg.cos } 30^\circ.$$

Effective force producing motion

= component of weight of block, parallel to slope — force of friction

$$= \text{Mg.sin } 30^\circ - \mu.\text{Mg.cos } 30^\circ$$

$$= \text{Mg.}\{{\cdot}500 - ({\cdot}364 \times {\cdot}866)\} \text{ poundals}$$

$$= \text{M}({\cdot}500 - {\cdot}315)32{\cdot}2 \text{ poundals}$$

$$= 5{\cdot}955.\text{M poundals.}$$

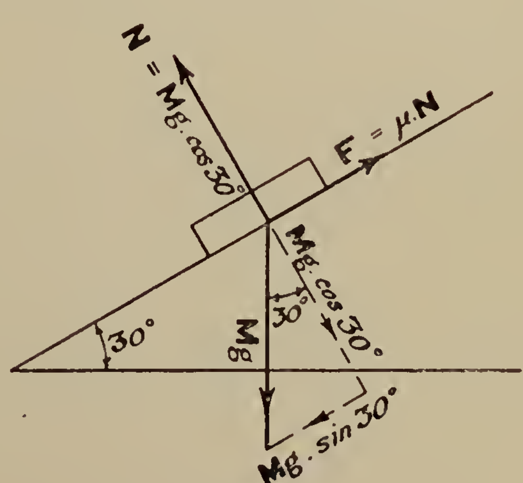


Fig. 73.

Acceleration of block

$$= \frac{\text{force producing motion}}{\text{mass of block}}$$

$$= \frac{5{\cdot}955 . \text{M poundals}}{\text{M pounds}}$$

$= 5{\cdot}955$ feet per second per second.

Let $t =$ time in seconds taken by block to slide 8 feet.

Velocity of block after t seconds from rest

$=$ acceleration $\times$ time

$= 5{\cdot}955$ feet per second per second $\times$ t seconds

$= 5{\cdot}955t$ feet per second.

Mean velocity $= \frac{1}{2}$ final velocity

$2{\cdot}978t$ feet per second.

Distance covered $=$ mean velocity $\times$ time

$= 2{\cdot}978t^2$ feet.

But this distance is 8 feet: therefore

$$2{\cdot}978\, t^2 = 8 \text{ feet,}$$

whence $$t^2 = \frac{8}{2{\cdot}978}$$

$$= 2{\cdot}685$$

and $$t = 1{\cdot}64 \text{ seconds.}$$

5. *A six-coupled goods locomotive, weighing* 44 *tons, draws a train weighing* 580 *tons. If the coefficient of friction between the wheels and the rails is* ·085 *and the tractive resistance is* ($\frac{1}{4}V + 2$) *pounds-weight per ton, what is the maximum speed V, in miles per hour, attainable on the level?*

Maximum tractive force exerted by the locomotive (all the wheels being driving-wheels)

$=$ weight of locomotive $\times$ coefficient of friction

$= 44$ tons-weight $\times$ ·085

$= 3{\cdot}74$ tons-weight

$= 8{,}380$ pounds-weight.

Tractive resistance of train

$= 580$ tons $\times$ ($\frac{1}{4}$V $+ 2$) pounds-weight per ton

$=$ (145V $+$ 1,160) pounds-weight.

When the speed is a maximum the tractive resistance will equal the maximum tractive force exerted by the locomotive, i.e.,

$$145V + 1{,}160 = 8{,}380,$$

whence
$$145.V = 8{,}380 - 1{,}160$$
$$= 7{,}220$$

and
$$V = \frac{7{,}220}{145}$$
$$= 49{\cdot}8 \text{ miles per hour.}$$

6. *A ladder, 24 feet in length, rests with its upper end against a wall at an angle of 30 degrees. Neglecting the weight of the ladder, find how far a man can ascend before the ladder begins to slip. Coefficient of friction between wall and ladder = $\frac{1}{3}$. Coefficient of friction between ladder and ground = $\frac{1}{4}$.*

First we have to make a sketch and mark on it the information given (see Fig. 74).

When the ladder is on the point of slipping, the forces acting upon it are :—

(*a*) the normal (horizontal) reaction, N_1, of the wall ;

(*b*) the force of friction, F_1 ($=\mu_1 N_1$), between the wall and the ladder, opposing motion and therefore vertically upwards ;

(*c*) the normal (vertical) reaction, N_2, of the ground ;

(*d*) the force of friction, F_2 ($=\mu_2 N_2$), between the ground and the ladder, opposing motion and therefore horizontally towards the wall ;

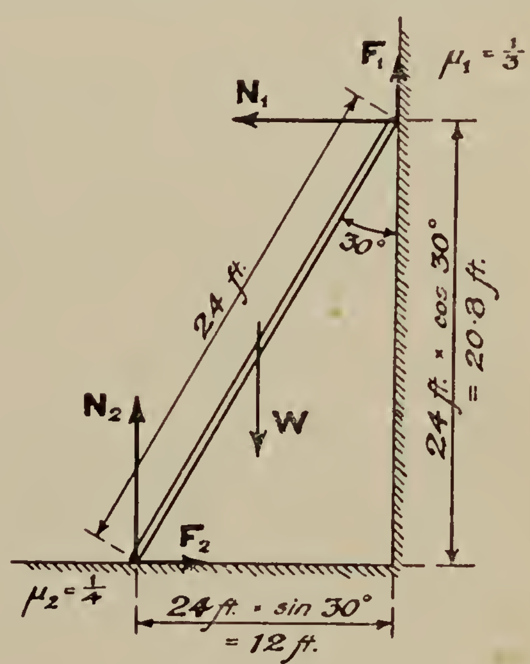


FIG. 74.

and (*e*) the weight, W, of the man, acting vertically downwards.

The forces N_1 and F_1 at the upper end of the ladder may be compounded into a single force R_1 in a direction which makes an angle ϕ_1 with the horizontal, such that :—

$$\tan \phi_1 = \mu_1 = \tfrac{1}{3}. \qquad \text{(see Fig. 75)}$$

Similarly, the forces N_2 and F_2 at the lower end of the ladder, may be compounded into a single force R_2 in a direction, which makes an angle ϕ_2 with the vertical, such that :—

$$\tan \phi_2 = \mu_2 = \tfrac{1}{4}.$$

We have now reduced the forces acting upon the ladder to three,

viz., R_1, R_2, and W. As the ladder is in equilibrium under the action of these three forces, and they are not all parallel, therefore they must all meet at one point, say C. This point must evidently be the intersection of the lines of action of the forces R_1 and R_2, so that the weight of the man must also act through C.

By determining the horizontal distance of C from the foot of the ladder, we can find the line of action of the weight of the man when

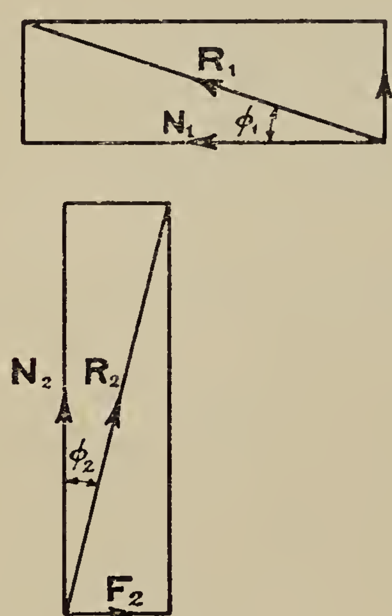


FIG. 75.

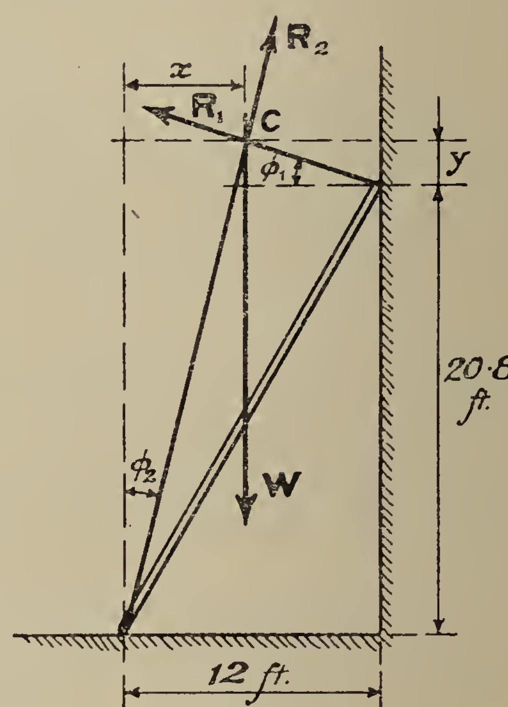


FIG. 76.

the ladder is upon the point of slipping, and hence can determine how far it is possible for him to ascend.

Let x = horizontal distance of the point C from the foot of the ladder, in feet.

Let y = vertical distance of the point C from the top of the ladder, in feet.

Then we have :—

$$\tan \phi_1 = \frac{y}{12 \text{ feet} - x} = \mu_1 = \tfrac{1}{3},$$

whence $$3y = 12 \text{ feet} - x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and $$\tan \phi_2 = \frac{x}{20{\cdot}8 \text{ feet} + y} = \mu_2 = \tfrac{1}{4},$$

whence $$20{\cdot}8 \text{ feet} + y = 4x \qquad \text{and}$$

$$y - 4x = -20{\cdot}8 \text{ feet} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Multiplying (2) by 3, we have :—

$$3y - 12x = -62{\cdot}4 \text{ feet}$$

also $$3y + x = 12 \text{ feet} \qquad \text{from (1) above :}$$

whence, by subtraction,

$$13x = 74{\cdot}4 \text{ feet}$$

and $$x = 5{\cdot}72 \text{ feet.}$$

Therefore the man can ascend the ladder to a fraction of the height equal to :—

$$\frac{x}{12 \text{ feet}} = \frac{5{\cdot}72 \text{ feet}}{12 \text{ feet}}.$$

Measuring up the ladder, therefore, he can ascend

$$\frac{5{\cdot}72}{12} \times 24 \text{ feet} = 11{\cdot}44 \text{ feet.}$$

7. *A railway train of* 240 *tons mass is hauled by a locomotive which exerts a draw-bar pull of* 3 *tons-weight. If the mean tractive resistance is* 13 *pounds-weight per ton, how long will be occupied in attaining a speed of* 36 *miles per hour, and what distance will be travelled in that time?*

Effective pull on train to produce acceleration
= draw-bar pull − tractive resistance
= (3 × 2,240) pounds-weight − (240 × 13) pounds-weight
= 6,720 pounds-weight − 3,120 pounds-weight
= 3,600 pounds-weight
= (3,600 × 32·2) poundals
= 116,000 poundals.

Acceleration of train

$$= \frac{\text{force producing acceleration}}{\text{mass of train}}$$

$$= \frac{116{,}000 \text{ poundals}}{(240 \times 2{,}240) \text{ pounds}}$$

$$= {\cdot}2158 \text{ foot per second per second.}$$

Time occupied in attaining given velocity

$$= \frac{\text{velocity}}{\text{acceleration}}.$$

In this case the velocity = 36 miles per hour = $36 \times \frac{22}{15}$ feet per second = 52·8 feet per second.

Therefore time occupied

$$= \frac{52{\cdot}8 \text{ feet per second}}{{\cdot}2158 \text{ foot per second per second}}$$

$$= 245 \text{ seconds}$$

$$= 4\tfrac{1}{12} \text{ minutes.}$$

Distance travelled

$$= \text{mean velocity} \times \text{time}$$
$$= \tfrac{1}{2} \times 36 \text{ miles per hour} \times \frac{4\frac{1}{12}}{60} \text{ hour}$$
$$= \frac{36 \times 49}{2 \times 60 \times 12} \text{ miles}$$
$$= 1{\cdot}225 \text{ miles.}$$

8. A force of 37 poundals is required to make a mass of $4\frac{1}{2}$ pounds slide on a horizontal surface. What is the coefficient of friction for this pair of surfaces?

9. If the coefficient of friction between a body of 12·3 pounds mass, and the horizontal surface on which it rests, is ·31, find the force, in F.P.S. units, required to make the body slide.

10. What is the mass of a body, in F.P.S. units, when the force required to move it horizontally is 358 F.P.S. units, and the coefficient of friction is ·217?

11. A body of 14 pounds mass rests on an inclined plane. When the angle of inclination is increased to $18\frac{1}{2}$ degrees, the body slides down the plane. Find (*a*) the force of friction, and (*b*) the normal reaction between the plane and the body.

12. What force, in C.G.S. units, must be applied to a block of wood, of 3·74 kilogrammes mass, in order to make it slide on a table, if the coefficient of friction between the block and the table is ·245?

13. The mass of a slab of iron in F.P.S. units is numerically equal to one-seventh of the force in poundals required to move it horizontally on a bench. What is the coefficient of friction between the slab and the bench?

14. A block of wood rests on a plane inclined at an angle of 12 degrees from the horizontal. If the mass of the block is 6·54 pounds, and the coefficient of friction between block and plane is ·28, determine the force required, parallel to the plane, to make the block slide down.

15. Calculate the coefficient of friction if a one-pound slab of lead, resting on a plane inclined at an angle of 15 degrees from the horizontal, requires a force of 16 poundals, parallel to the plane, to make it slide up the plane.

16. A block of wood of 28 pounds mass carries a casting of $3\frac{1}{4}$ cwt. mass. If the coefficient of friction between the block and the floor is ·37, what force, in pounds-weight, will be required in order to drag the block along the floor?

17. What draw-bar pull must a railway locomotive exert, when hauling a train of 198 tons mass, in order to attain a speed of 53 miles

per hour in $2\frac{1}{2}$ minutes from rest? The mean value of the tractive resistance may be taken as 13 pounds-weight per ton.

18. A wooden box rests on a plank, and it is found that, if the plank is inclined at an angle of 73 degrees to the vertical, the box just starts to slide down the plank. How far down the plank will the box slide when the angle is decreased to 65 degrees, if the time occupied is $2\frac{1}{4}$ seconds?

19. A railway locomotive draws a train weighing 380 tons, including the engine itself, which weighs 56 tons. If the coefficient of friction between the wheels and the rails is ·08 and the tractive resistance is $(\frac{1}{4}V + 2)$ pounds-weight per ton, determine the maximum speed V, in miles per hour, which the train can attain on a level line.

20. A uniform ladder, weighing one hundredweight, and 30 feet in length, rests with its lower end on level ground and its upper end against a wall. If the coefficient of friction between the ladder and the wall is ·35 and that between the ladder and the ground is ·28, determine the greatest angle which the ladder can make with the wall without slipping.

21. A man weighing 12 stone climbs a uniform ladder, which weighs 100 pounds, is 27 feet in length, and is placed against a wall at an angle of 30 degrees. How far can he ascend before the ladder begins to slip, if the coefficient of friction between the wall and the ladder is ·4 and that between the ground and the ladder is ·22? How far could a boy weighing $7\frac{1}{2}$ stone ascend?

22. A train weighing 300 tons is drawn by a locomotive which exerts a pull of 8,000 pounds-weight. Find the time occupied in attaining a speed of 45 miles per hour from rest, the distance travelled in that time, and the momentum acquired by the train. Mean tractive resistance = ·0055 of the weight of the train.

23. A block of wood of $3\frac{3}{4}$ pounds mass rests on an inclined plane at an angle of 60 degrees from the horizontal, and is prevented from sliding down the plane by a light string parallel to the slope of the plane. If the coefficient of friction between the block and the plane is ·32, determine the tension in the string.

24. How far will a coal-truck, of 10 tons mass, travel on level ground, if it is given a velocity of 8 miles per hour, and the mean tractive resistance is 9·7 pounds-weight per ton? Find also how long it will take to come to rest.

25. A uniform ladder weighing 42 pounds leans against a smooth wall, at an angle of 30 degrees, and its lower end rests on a rough horizontal floor. Find the pressure on the wall, and the magnitude and direction of the force exerted by the ground upon the ladder. What is the minimum value of the coefficient of friction for the ladder and the floor, in order that the ladder may not slip?

CHAPTER 10: STRESS AND STRAIN

UP to the present we have considered only the action of **external** forces on rigid bodies. We must now think of what takes place *inside* a body when external forces act upon it.

Consider the case of a water-tank supported upon iron columns, as indicated in Fig. 77. The force of gravitation acts on the tank and, if there were no other force to oppose it, would draw the tank downwards towards the earth, that is to say, the tank would fall. Why does it not do so? Obviously, because the **weight** of the tank and its contents is balanced by the equal and opposite **reactions** from the supporting columns.

Fig. 77.

Now instead of thinking of the forces acting upon the *tank*, let us think of those acting upon the *columns*. These forces are the *weight* of the tank and its contents acting vertically downwards, and the equal and opposite *reactions* from the ground, acting vertically upwards. Each column is thus between opposing forces which tend to push it in opposite directions. These two forces counteract each other, but they can only do so through the medium of the column: they do not actually meet, since the weight of the tank acts upon the top surface of the column, and the reaction of the ground acts upon the base of the column.

There must, then, be a pair of **internal forces** in the material of the column, enabling it to resist the external forces acting upon it: otherwise it would collapse.

Clearly there must be one internal force acting *upwards* to balance the downward force exerted by the weight of the tank, and another internal force, of equal magnitude, acting *downwards* to balance the upward reaction of the ground, as shown in Fig. 78. Any such pair of internal forces in the material of a body we term a **stress.**

Here, again, we have a term which is used in Mechanics in a strictly limited and defined sense, whereas in everyday affairs it is used much more loosely. We must therefore, as in all such cases, take particular care that we do not employ the word in any other capacity than that given in the definition: *a stress is a pair of equal and opposite internal forces in the material of a body.*

Fig. 78.

CLASSIFICATION OF STRESSES. Corresponding to three different arrangements of external forces acting upon a body, we have three different kinds of simple stress produced within a body.

(i) Two forces, in the same straight line, **pulling** a body in opposite directions, produce what we term **tensile stress** in the material of the body, and the latter is said to be in a state of **tension.**

(ii) Two forces, in the same straight line, **pushing** a body in opposite directions, produce what we term **compressive stress** in the material of the body, and the latter is said to be in a state of **compression.**

(iii) Two pairs of forces, forming two equal and opposite **couples,** acting upon a body, produce what we term **shear stress** in the material of the body, and the latter is said to be in a state of **shear.**

Because the forces acting are all in the same straight line, tension and compression are both called **direct stresses,** whereas shear, being produced by the action of couples, is called a **tangential stress.** Let us consider each kind of stress separately.

TENSION. In this case the body is *pulled* in opposite directions by external forces, as indicated in Fig. 79. For example, the rope used in a "tug-of-war" is in tension, the teams pulling it in opposite directions. The cord supporting the lamp, shown in Fig. 46, is also in tension, the opposing forces being the weight of the lamp and the reaction of the ceiling. The traces by which a horse draws a cart give us yet another example of a body in tension.

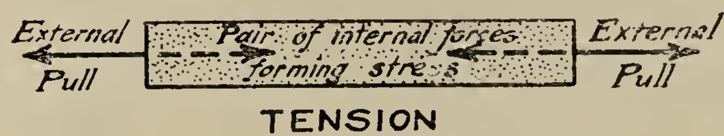


Fig. 79.

COMPRESSION. In this case the body is *pushed* in opposite directions by external forces, as indicated in Fig. 80. The columns shown in Figs. 77 and 78 are a good example of compressive stress. The opposing forces are, of course, the weight of the tank, and the reactions of the ground. The walls of a building are another similar example of compression. A further instance may be found in the legs of a chair or table. It should be noticed that, in order to withstand *compression*, a body must possess the quality of *stiffness* or *rigidity*: a rope, or a chain, or any similar flexible body is quite unsuited to take a compressive stress, although well adapted to withstand tension.

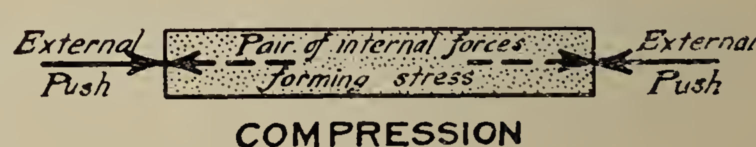


Fig. 80.

SHEAR. In this case the body is acted upon by two pairs of equal, unlike, parallel forces, that is, by two couples, as indicated in Fig. 81. A nail driven into a wall and carrying a picture is an example of this kind of stress, for the weight of the picture and the vertical reactions of the wall are parallel forces forming couples which act upon the nail. A rivet

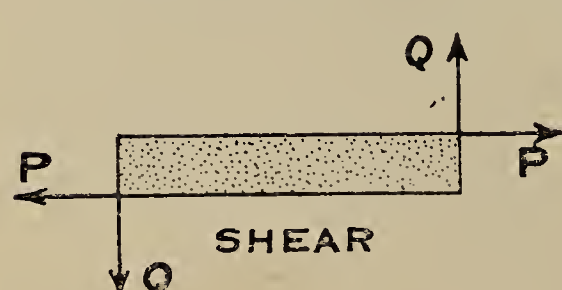


Fig. 81.

joining two plates, as shown in Fig. 82, gives us another instance of shear stress. Shear is so called from the action of a pair of shears, which exert this kind of stress.

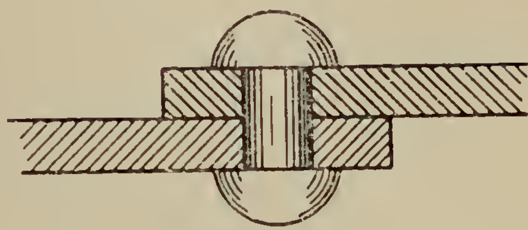


Fig. 82

There are other more complicated kinds of stress, which consist of combinations of simple stresses. Consideration of these must, however, be deferred for the present.

STRESS INTENSITY. As any stress is called into existence to balance external forces, it is evident that the magnitude of the stress will depend upon the magnitude of the external forces, and must, in fact, be equal thereto. A pull of, say, 3 pounds-weight on each end of a rope, for example, will produce a tension of 3 pounds-weight in the material of the rope. Similarly, if a column carries a load of, say, 50 tons-weight, then the magnitude of the compression in the material of the column will be 50 tons-weight. The magnitude of a stress is the magnitude of *each* of the two forces composing it, not of the two together.

It is quite evident that a small rope, or a small column, will not be as strong as a large one, even though it be made of exactly the same quality material. The *size* of a rope, or a column, or any other body affects its strength, as well as the nature of the material of which it is made.

If, then, we are considering the effect of the external forces acting upon a body in producing stresses in the material of which it is composed, we must certainly take into account the **cross-sectional area** of the body, that is, the area over which the stresses are distributed.

In general, it may be taken that simple stress acting in a body is evenly distributed throughout the material, and we measure the effect of the stress, accordingly, by the amount of stress acting upon each unit area of the cross-section. This stress per unit area is termed

the **stress intensity** in the material of the body, but in practice this is frequently abbreviated to *stress* simply.

Putting the same thing in slightly different words, we may define stress intensity as *the whole stress divided by the area on which it acts.*

It should be noted that the word *stress* is used both for the *whole* internal force in a piece of material, and also for the *intensity* of that force, or the force per unit area. It is therefore necessary, when using the term, to be quite clear in which sense it is employed, and to avoid any ambiguity in expression about it.

POSITIVE AND NEGATIVE STRESS. It is convenient sometimes to speak of one kind of direct stress as *positive*, and the other as *negative*. Thus, if we take compressive stress as positive, tensile stress will be negative, and vice versa. As this is merely a matter of convenience, there can be no absolute rule as to which shall be considered positive and which negative, and in any particular case it should be stated which arrangement has been adopted. In this book we shall employ the more usual convention of taking *compression as positive* and *tension as negative.*

DIMENSIONS OF STRESS. A stress (i.e., a *whole* stress) is simply a pair of forces in the material of a body, and therefore its dimensions will be the same as those of any other force, that is, $[M][L][T]^{-2}$. Stress *intensity*, often called simply "stress," is the stress per unit area, that is, the whole stress divided by the area of the cross-section upon which it acts. The dimensions of stress intensity are therefore the dimensions of a stress, $[M][L][T]^{-2}$, divided by the dimensions of an area, viz. $[L]^2$, that is, they are $[M][L]^{-1}[T]^{-2}$. The dimensions of a stress are not affected by the nature of the stress, that is, they are just the same for tension compression, or shear.

STRAIN. When a piece of material of any kind is pulled it stretches, though the amount of stretch may not

be great, and, indeed, may not be visible to the naked eye.

Similarly, when a piece of material is pushed it shortens, and when it is acted upon by a couple its shape is changed, though in either case the change may be very small.

Any such change of size or shape is termed a **strain.**

It is found by experiment that **stress** in a material is always accompanied by **strain.** We know of no material of which this is not true.

When a piece of material undergoes tensile stress, then it also suffers tensile strain or extension.

When a piece of material undergoes compressive stress, then it also suffers compressive strain or shortening.

When a piece of material undergoes shear stress, then it also suffers shear strain or angular distortion.

Just as we saw that stress is used in two slightly different ways, so we have to notice that strain may be used in two different ways. We may speak of the strain suffered by a body when we mean the *total* change of size or shape it has undergone, or we may use the word strain to denote the *fractional* change of size or shape. It is preferable to limit the use of *strain* to the latter meaning, and to term the total change the *extension, shortening,* or *distortion,* as the case may be. For example, in the case of tension, we shall regard the strain as the *ratio* of the total extension of the material to the original length, i.e. :—

$$\text{Strain} = \frac{\text{total extension}}{\text{o riginallength}}.$$

DIMENSIONS OF STRAIN. Strain may be defined in the case of tension or compression as the *change of length divided by the original length.* The dimensions of a strain are therefore **[L]** divided by **[L]**, that is $[\mathbf{L}]^0$ or **1**, so that a strain is a purely numerical quantity. In the case of shear, the strain is the *angle of distortion,* measured in radians. An angle in circular measure is

equal to arc divided by radius, so that its dimensions must be **[L]** divided by **[L]**, that is $[\mathbf{L}]^0$ or **1**, as before. In all cases, therefore, the dimensions of a strain are **1**, i.e., it is a numerical quantity only.

HOOKE'S LAW. It will be seen that *stress* and *strain* are intimately connected with each other, and, indeed, that we cannot have the one without the other. It is essential, however, that we should be perfectly clear as to the distinction between them. In everyday life the two terms are used more or less interchangeably, but in Mechanics they must on no account be confused. Let us remind ourselves, then, that a *stress* is a pair of internal forces in the material of a body, whereas a *strain* is a change of size or shape accompanying a stress.

Not only are stress and strain closely connected with one another, but within certain limits their magnitudes are proportional. This was discovered by Robert Hooke, an English experimental philosopher, in the latter part of the seventeenth century, and is known accordingly as **Hooke's Law.**

Hooke's Law is best stated in the form *strain is proportional to stress.* This law is of very wide application, but it is not universally true. Most materials obey it, more or less exactly, provided that the stress intensity is not greater than some particular value, which depends upon the nature of the material and is known as the **elastic limit** for that material. There are, however, some materials which do not obey Hooke's Law at all.

We must take care to remember, in considering this law, that it refers to *fractional* strain and *intensity* of stress: never to total stress or total strain.

ELASTIC MODULI. It follows from Hooke's Law, that, for any given material which obeys that law, and for a particular kind of stress, the ratio: $\dfrac{\text{stress}}{\text{strain}}$ is a constant. Such a constant is termed a **Modulus of Elasticity.**

With most materials it is found that, within the

limits of the proportionality of stress to strain, on the removal of the stress the strain also disappears. This property of a material is known as **elasticity**: hence the terms **elastic limit** and **modulus of elasticity.**

For many materials the modulus of elasticity has the same value for either kind of *direct* stress, i.e., for either tension or compression. For either of these stresses, therefore, the modulus is known by the same name, and is called **Young's Modulus** after Thomas Young, an English physician and natural philosopher who did much valuable work in the early part of the nineteenth century.

In the case of shear stress the modulus has quite a different value, and is known as the **Modulus of Rigidity.**

As these constants occur very frequently in scientific and engineering calculations, it is a convenience to have recognised symbols for them: accordingly we denote Young's Modulus by the letter **E**, and the Modulus of Rigidity by the letter **G.**

DIMENSIONS OF MODULI OF ELASTICITY. In all cases a modulus of elasticity is the ratio of *stress intensity* to the corresponding *fractional strain.* Its dimensions are therefore those of a stress intensity, $[M][L]^{-1}[T]^{-2}$, divided by those of a strain, **1**, that is, they are the same as the dimensions of a stress intensity. This is true of both Young's Modulus and the Modulus of Rigidity.

SHEAR AND TORSION. Torsion, or twisting, is a particular case of simple shear. Let us see if we can realise the difference between torsion and other cases of simple shear.

Imagine that we have a number of wooden discs, all of the same size, and that they are all placed together and holes bored through them, parallel to the axis of the cylinder so formed, as shown in Fig. 83.

Now let us suppose that we fill up these holes with some material softer than the wood of which the discs

are composed. It will help to clarify our ideas if we imagine that we insert a wax candle in each hole, so as just to fill it.

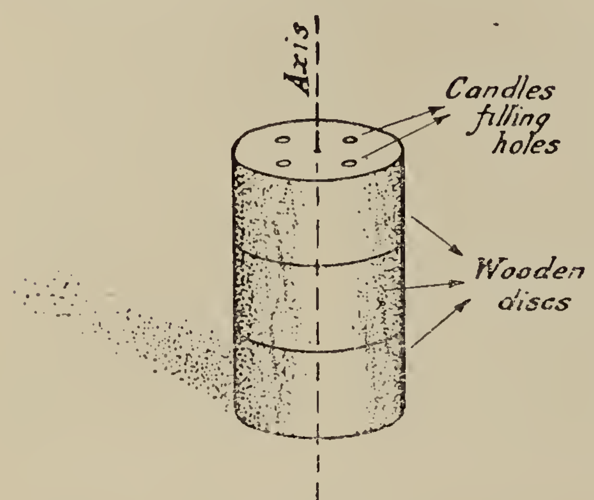
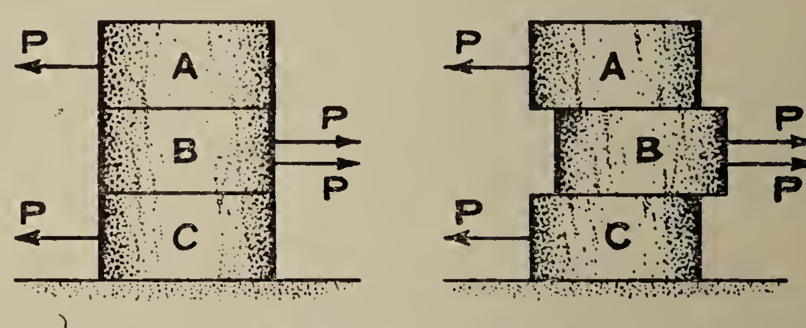


Fig. 83.

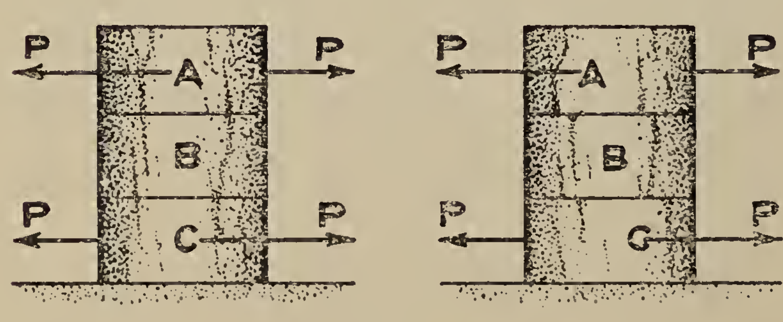
Now if we apply parallel forces, of equal magnitude P, to each of the discs A and C, as shown in Fig. 84, and two forces equal and opposite to the first two to the middle disc B, these four forces will constitute two equal and opposite couples *in the same plane*. These couples will put the material of the candles in a state of simple shear. If the shear stress induced is great enough, the candles will be actually cut through and the disc B will move across the neighbouring discs A and C, as shown. This is an approximate demonstration of what we mean by **simple shear** in ordinary cases.



SIMPLE SHEAR

Fig. 84.

If, instead, we apply equal and opposite couples to the discs A and C, the couples acting *in parallel planes* perpendicular to the axis of the cylinder, as shown in Fig. 85, then these couples will tend to twist the cylinder about its axis. This will again put the material of the candles in a state of simple shear. If the shear stress induced is great enough, the candles will in this case also be actually cut through. But this time

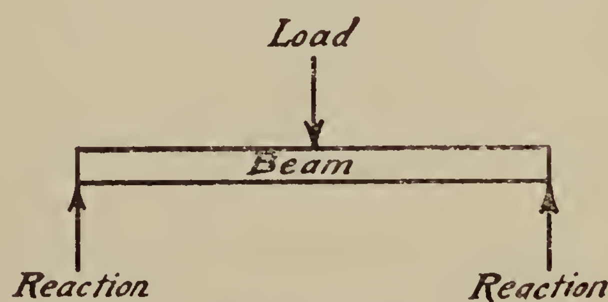


TORSION

Fig. 85.

the discs, instead of simply sliding across each other, as in the previous case, will rotate across each other round the axis of the cylinder: i.e., the motion will be one of *rotation* instead of *translation*. This is an approximate demonstration of what we mean by **torsion**: it will now readily be seen that torsion is a special case of shear.

BENDING STRESS. When we *bend* a piece of material, the couples which we apply to the body are resisted by stresses set up in the material. These stresses consist partly of tension, partly of compression, and partly of shear, and are too complicated for us to deal with here. They are of great importance in the theory of structures, since **beams**, which form so large a part of many structures, are simply bars of material subjected to **bending moments**, that is, to couples which produce bending.



BENDING

FIG. 86.

We may, however, notice that the *strain* is always proportional to the *stress intensity*, provided that the elastic limit of the material is not exceeded, just as in cases of simple stress. Also, the *deflection* of the beam is proportional to the *load* it carries (see Fig. 86).

COMBINED STRESSES. A piece of material may be subjected to external forces which will produce in it stresses of more than one kind at the same time. For example, a column may be subjected to forces which will produce not only compression but also bending, and a shaft used for transmitting power to machinery may be subjected to forces which will produce in it both torsion and bending.

All kinds of stress, however complicated they may be, can be shown to be due merely to the combination, in various ways, of *simple stresses*, by which term we denote *tension*, *compression*, and *shear*.

STRESSES IN FRAMED STRUCTURES. Instead of using simple beams and columns, we frequently employ **framed structures**. We may define a framed structure as one which is composed of a number of bars joined together at their ends, so as to form a frame of definite shape, and, usually, so that each bar in the frame shall be subjected mainly to *direct stress*. The subject of framed structures is too big to be dealt with in detail here, but we may well devote a little attention to some of the more important principles involved.

We usually make our frames in such a way that they are composed of **triangles**, because the triangle is the only simple figure of which the lengths of the sides are sufficient to fix the shape. For example, the **Warren girder**, sketched in Fig. 87, has its members so arranged as to form equilateral triangles.

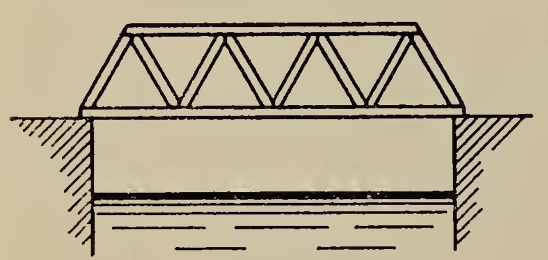


FIG. 87.

Further, in considering the stresses set up in the bars which form the frame, we assume that all the joints are **hinged** and free from friction. These assumptions may not correspond with the actual facts, but they are very useful, for they mean that we can take all the stresses in the bars as being either **pure tension** or **pure compression**. The errors involved in these assumptions are not usually serious, and are, in general, on the side of additional safety.

The general method employed to determine the magnitude of the stresses in a framed structure consists of the application of the **Triangle of Forces** to each of the joints of the structure in turn. All the triangles so obtained are usually combined in a single diagram which is termed the **stress diagram** for the structure. Students who are not thoroughly familiar with that very important law, the Triangle of Forces, should turn back to Chapter 4 and revise their knowledge of it.

Take the case of the simple **roof truss** shown in Fig. 88. This is a frame consisting of steel bars joined

together at their extremities, so as to form a continuous series of triangles, and is used to support the roof over a building. The external forces P_1, P_2, and P_3 are due mainly to the weight of the roof. These forces are balanced by the reactions R_1 and R_2, at the points of support, so that the five external forces are in equilibrium amongst themselves.

Let us suppose that we are given the size and form of the roof truss and the magnitude and position of each of the loads upon it. How are we to find the stresses in the various bars?

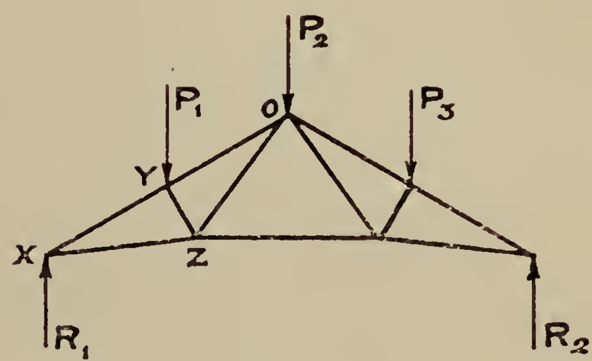


FIG. 88.

First we must determine the magnitude of the reactions R_1 and R_2. We can do this by means of the *Principle of Moments*, as explained in Chapter 3 (page 51). Having obtained these, we are in a position to draw our first Triangle of Forces.

Consider the joint X. Three forces act at this point, viz., the *reaction* R_1, the *stress* in the bar XY, and the *stress* in the bar XZ. Since these three forces meet at the point X and are in equilibrium, they can be represented both in magnitude and direction by the sides of a triangle taken in order. Drawing a vertical line, or *vector* (AB, Fig. 89) to represent the force R_1 to some convenient scale, we can draw through its ends lines parallel to the directions of the stresses in the bars XY and XZ, and so complete the triangle of forces, ABC, for the point X. Measuring the sides of this triangle, we obtain the magnitude of the stresses in the bars XY and XZ, and also their *sense*.

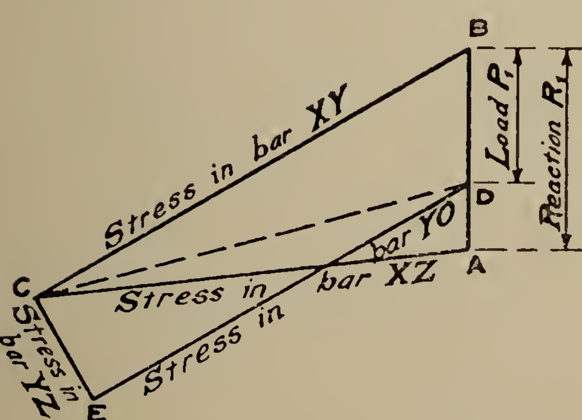


FIG. 89.

Now, having obtained the magnitude of the stress in the bar XY, and knowing the magnitude and direction of the force P_1, we can combine these two forces to give a single resultant. To do this we make BD equal to the force P_1 (to the scale already employed), and drawing CD, the latter will represent the required resultant. We then have at the point Y only three forces acting, viz., the resultant of the stress in XY and the force P_1, as just determined, the stress in the bar YZ, and the stress in the bar YO.

We can therefore draw the Triangle of Forces, CDE, for the point Y, and so determine the magnitude and sense of the stresses in the bars YZ and YO. Next we can apply the same method to the point Z, and then to the point O.

In this way we can go on from point to point and eventually determine the stresses in all the bars composing the frame. The method will be made much clearer by a careful study of the example worked out in detail, following this chapter.

POINTS TO REMEMBER. In dealing with framed structures and their stress diagrams, the following considerations should always be borne in mind:—

(*a*) The external forces acting on a frame must always be in equilibrium amongst themselves.

(*b*) All the joints are assumed to be frictionless hinges, so that the stresses are always either pure tension or pure compression. A bar which is subjected to compressive stress is termed a **strut**, and a bar which is subjected to tensile stress is termed a **tie**. All the bars in a frame are therefore assumed to be either struts or ties.

(*c*) Since a stress consists of a *pair* of forces, equal and opposite, it follows that the stress in a bar will act in opposite directions at the opposite ends of the bar. (Refer to Figs. 79 and 80.)

(*d*) At every stage of the work we must consider the forces acting at one of the joints and at *one only*. Con-

centrate attention on *one* joint at a time and confusion will be avoided.

(*e*) Remember that the "stress diagram" is merely a series of triangles of forces, one for each of the joints, combined in a single diagram for convenience.

(*f*) Results obtained graphically should be checked, whenever possible, by some other means.

Students should take every opportunity of noticing the form of any roof truss or bridge girder they may come across, and should observe how it is divided into triangles.

SUMMARY OF CHAPTER 10

A **stress** is a pair of equal and opposite internal forces in the material of a body.

Tension is the stress produced in a body when it is *pulled* in opposite directions.

Compression is the stress produced in a body when it is *pushed* in opposite directions.

Shear is the stress produced in a body when it is acted upon by two equal and opposite couples.

Stress intensity is the amount of stress per unit area of cross-section of a body.

Strain is the fractional change of size or shape of a body subjected to stress. **Tensile strain** is the ratio of the increase of length of a body, under tension, to its original length. **Compressive strain** is the ratio of the decrease of length of a body, under compression, to its original length. **Shear strain** is the *angle* (in radians) through which a body is distorted under shear stress.

Hooke's Law states that strain is proportional to stress, within certain limits.

The **elastic limit** for a material is the maximum intensity of stress for which the material obeys Hooke's Law. Within the elastic limit the strain disappears when the stress is removed.

A **modulus of elasticity** is the constant ratio of stress intensity to strain, for a particular material, under a particular kind of stress, and within the elastic limit. The modulus of elasticity for tension or compression is known as **Young's Modulus,** and is denoted by **E.** The modulus of elasticity for shear is known as the **Modulus of Rigidity** and is denoted by **G.**

Torsion is a particular case of shear stress produced by twisting a body. **Bending** is a combination of simple stresses.

The dimensions of a (total) stress are the same as those of any other force. The dimensions of a stress intensity are $[M][L]^{-1}[T]^{-2}$.

The dimensions of a modulus of elasticity are the same as those of a stress intensity. Strain is a purely numerical quantity.

Framed structures are usually composed of bars arranged so as to form triangles. The joints are assumed to be **frictionless hinges,** and the stresses in the bars are determined by drawing the Triangle of Forces for each joint in turn, thus producing the **stress diagram.**

EXAMPLES X

(*For Hints on Working Examples, see page* 21.)

1. *A rod of mild steel,* $1\frac{1}{2}$ *inches in diameter, and* 12 *feet in length, carries a load of* 13·2 *tons-weight. Calculate* (*a*) *the stress intensity,* (*b*) *the strain, and* (*c*) *the extension. Take E as* 13,000 *tons-weight per square inch.*

$$\text{Stress intensity} = \frac{\text{load}}{\text{cross-sectional area}}$$

$$= \frac{13\cdot 2 \text{ tons-weight}}{\frac{1}{4}\pi(1\frac{1}{2})^2 \text{ square inches}}$$

$$= \frac{13\cdot 2}{1\cdot 767} \text{ tons-weight per square inch}$$

$$= 7\cdot 47 \text{ tons-weight per square inch.}$$

$$\text{Strain} = \frac{\text{stress intensity}}{\text{Young's modulus}}$$

$$= \frac{7\cdot 47 \text{ tons-weight per square inch}}{13{,}000 \text{ tons-weight per square inch}}$$

$$= \cdot 000575.$$

$$\text{Extension} = \text{original length} \times \text{strain}$$

$$= 144 \text{ inches} \times \cdot 000575$$

$$= \cdot 0827 \text{ inch.}$$

2. *If E for mild steel is* 30,000,000 *pounds-weight per square inch, and the elastic limit is* 35,000 *pounds-weight per square inch, what is the maximum elastic extension possible in a flat bar,* $1\frac{1}{2}$ *inches wide,* $\frac{1}{2}$ *inch thick, and* 20 *feet in length, and what will then be the whole stress in the bar?*

$$\text{Maximum elastic strain} = \frac{\text{elastic limit}}{\text{Young's modulus}}$$

$$= \frac{35{,}000 \text{ pounds-weight per square inch}}{30{,}000{,}000 \text{ pounds-weight per square inch}}$$

$$= \frac{7}{6{,}000}.$$

$$\begin{aligned}\text{Max. elastic extension} &= \text{original length} \times \text{strain} \\ &= 240 \text{ inches} \times 7/6{,}000 \\ &= \cdot 357 \text{ inch.}\end{aligned}$$

$$\begin{aligned}\text{Whole stress} &= \text{stress intensity} \times \text{cross-sectional area} \\ &= 35{,}000 \text{ pounds-weight per square inch} \times \\ &\qquad (1\tfrac{1}{2} \times \tfrac{1}{2}) \text{ square inches} \\ &= 26{,}250 \text{ pounds-weight} \\ &= 11 \cdot 7 \text{ tons-weight.}\end{aligned}$$

3. *A tie-rod* 7 *feet in length is* $1\frac{1}{4}$ *inches in diameter and carries a load of* 8·4 *tons-weight. The extension produced is* ·043 *inch. Determine the stress intensity, the strain, and the value of Young's modulus for the material of the rod.*

$$\begin{aligned}\text{Stress intensity} &= \frac{\text{load}}{\text{cross-sectional area}} \\ &= \frac{8 \cdot 4 \text{ tons-weight}}{\frac{1}{4}\pi\,(1\frac{1}{4})^2 \text{ square inches}} \\ &= \frac{8 \cdot 4 \text{ tons-weight}}{1 \cdot 228 \text{ square inches}} \\ &= 6 \cdot 76 \text{ tons-weight per square inch.}\end{aligned}$$

$$\begin{aligned}\text{Strain} &= \frac{\cdot 043 \text{ inch}}{84 \text{ inches}} \\ &= \cdot 000512.\end{aligned}$$

$$\begin{aligned}\text{Young's modulus} &= \frac{\text{stress intensity}}{\text{strain}} \\ &= \frac{6 \cdot 76 \text{ tons-weight per square inch}}{\cdot 000512} \\ &= 13{,}200 \text{ tons-weight per square inch.}\end{aligned}$$

4. *What load can be carried by a cast-iron column of hollow circular cross-section,* 6 *inches external diameter and* 4 *inches internal diameter, if the allowable stress intensity is* 2·65 *tons-weight per square inch?*

$$\begin{aligned}\text{Cross-sectional area} &= \tfrac{1}{4}\pi\left\{(6)^2 - (4)^2\right\} \\ &= \tfrac{1}{4}\pi\,(36 - 16) \\ &= 5\pi \\ &= 15 \cdot 71 \text{ square inches.}\end{aligned}$$

Safe load = cross-sectional area × allowable stress intensity
= 15·71 square inches × 2·65 tons-weight per square inch
= 41·7 tons-weight.

5. *Find the force necessary to compress a cube of copper of* 2 *inches edge, through* ·001 *inch. E for copper is* 7,000 *tons-weight per square inch.*
If the crushing strength of copper is 24 *tons-weight per square inch, what is the factor of safety?*

$$\text{Strain} = \frac{\text{shortening}}{\text{original length}}$$
$$= \frac{\cdot 001 \text{ inch}}{2 \text{ inches}}$$
$$= \cdot 0005.$$

$$\text{Stress intensity} = \text{Young's modulus} \times \text{strain}$$
$$= 7{,}000 \text{ tons-weight per square inch} \times \cdot 0005$$
$$= 3\cdot 5 \text{ tons-weight per square inch.}$$

$$\text{Force required} = \text{stress intensity} \times \text{cross-sectional area}$$
$$= 3\cdot 5 \text{ tons-weight per square inch} \times 4 \text{ square inches}$$
$$= 14 \text{ tons-weight.}$$

$$\text{Factor of safety} = \frac{\text{crushing stress}}{\text{actual stress}}$$
$$= \frac{24 \text{ tons-weight per square inch}}{3\cdot 5 \text{ tons-weight per square inch}}$$
$$= 6\cdot 85.$$

6. *A roof-truss, of* 36 *feet span and* 30 *degrees slope, is loaded as shown in the sketch. Draw the stress diagram for the truss, and tabulate the stresses in the members, indicating which are in tension and which are in compression.*

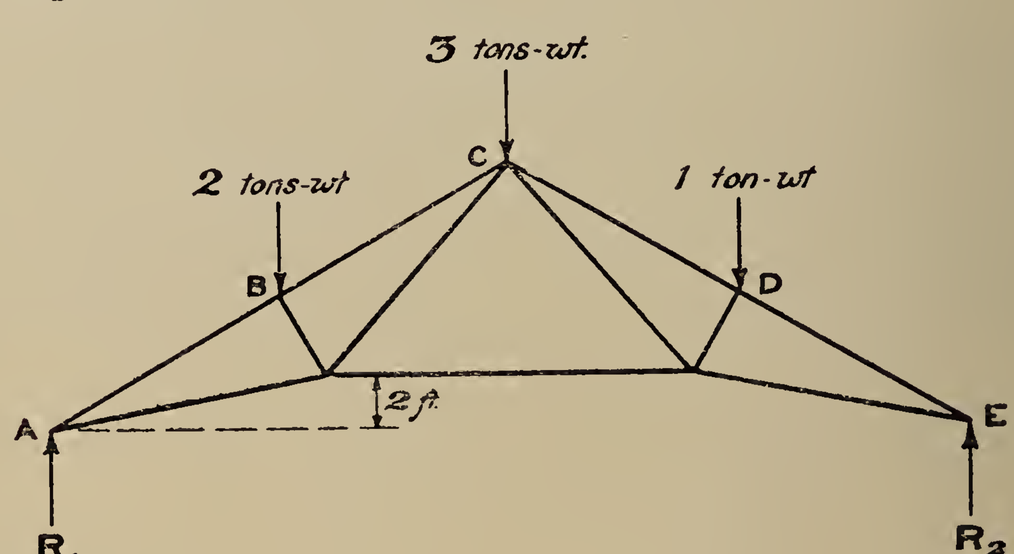


FIG. 90.

The first thing to do is to draw out the "Frame Diagram" (as sketched in the question) as accurately as possible, and to as large a scale as we can conveniently use (Fig. 91).

Next we letter the *spaces* of the frame diagram ("Bow's Notation"), using one letter only for each space between two adjacent external forces, and one for each triangular space in the frame.

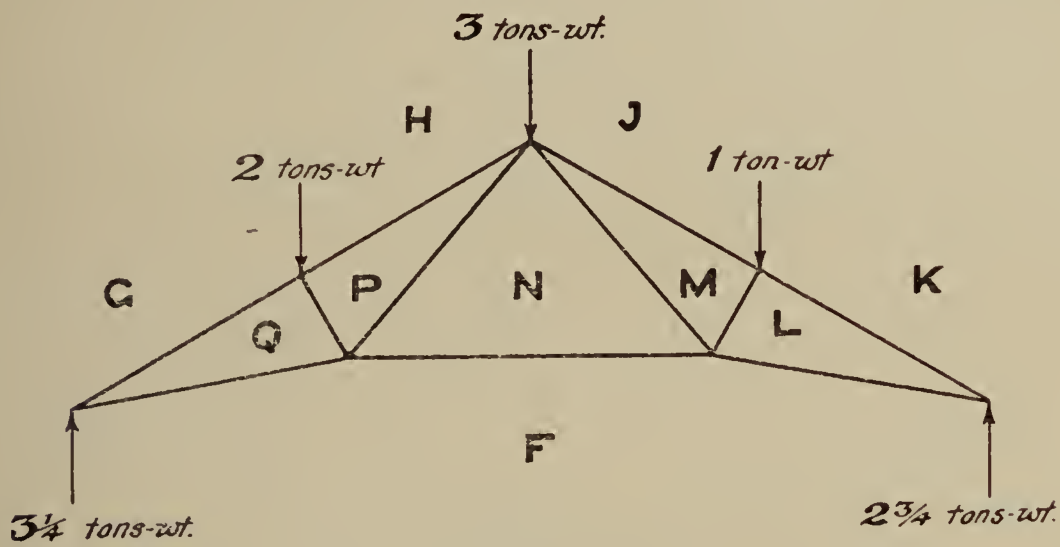


FIG. 91.

We now have to determine the magnitudes of the reactions, R_1 and R_2, at the points of support. As all the external forces, including the reactions, are vertical, we can very readily do this by taking moments about one of the points of support, say A. Let S = span of truss in feet. Then we have :—

$$R_2 \text{ tons-weight} \times S \text{ feet} = (2 \text{ tons-weight} \times \tfrac{1}{4}.S) + (3 \text{ tons-weight} \times \tfrac{1}{2}.S) + (1 \text{ ton-weight} \times \tfrac{3}{4}.S),$$

whence $R_2 = \frac{1}{2} + 1\frac{1}{2} + \frac{3}{4} = 2\frac{3}{4}$ tons-weight,

and $R_1 = 2 + 3 + 1 - 2\frac{3}{4} = 3\frac{1}{4}$ tons-weight.

We can now commence the "stress diagram" by setting out the external forces to a suitable scale (as large as convenient). The downward external forces are : GH = 2 tons-weight, HJ = 3 tons-weight, and JK = 1 ton-weight. These we represent by the lines *gh*, *hj*, and *jk*, parallel and proportional to the corresponding forces (Fig. 92).

Next we have to deal with the reactions R_1 and R_2, which we now call FG and KF in accordance with the notation we have adopted. Using the same scale, we measure *kf* to represent KF = $2\frac{3}{4}$ tons-weight upward from *k*. This gives us the point *f*. Then *fg* represents, to the same scale, the reaction FG = $3\frac{1}{4}$ tons-weight.

Now consider the forces acting at the point A. They are : the reaction FG = $3\frac{1}{4}$ tons-weight, the stress in the bar GQ, and the stress in the bar QF. We know the *directions* of all three forces, but the

magnitude of the first only. We can find the magnitudes of the other two forces by drawing the triangle of forces, since the forces are in equilibrium. The reaction FG is already represented in magnitude and direction by the line *fg*, so we obtain our triangle by drawing through the point *g* a line parallel to the bar GQ, and through the point *f* a line parallel to the bar QF: these lines intersect at the point *q*, so that *fgq* is the triangle of forces for the point A, *gq* representing the force in the bar GQ, and *qf* the force in the bar QF.

Now consider the forces acting at the point B. Here we know the forces QG and GH, both in magnitude and direction: they are represented on the stress diagram by *qg* and *gh*, so that their resultant is represented by the dotted line *qh*. We may therefore regard the forces at B as three only, viz., QH (the resultant of QG and GH), HP, and PQ. These three are in equilibrium, and we can therefore draw the corresponding triangle of forces, by drawing *hp* parallel to HP through *h*, and *qp* parallel to PQ through *q*, to meet in *p*. Then *hpq* is the triangle of forces for the point B.

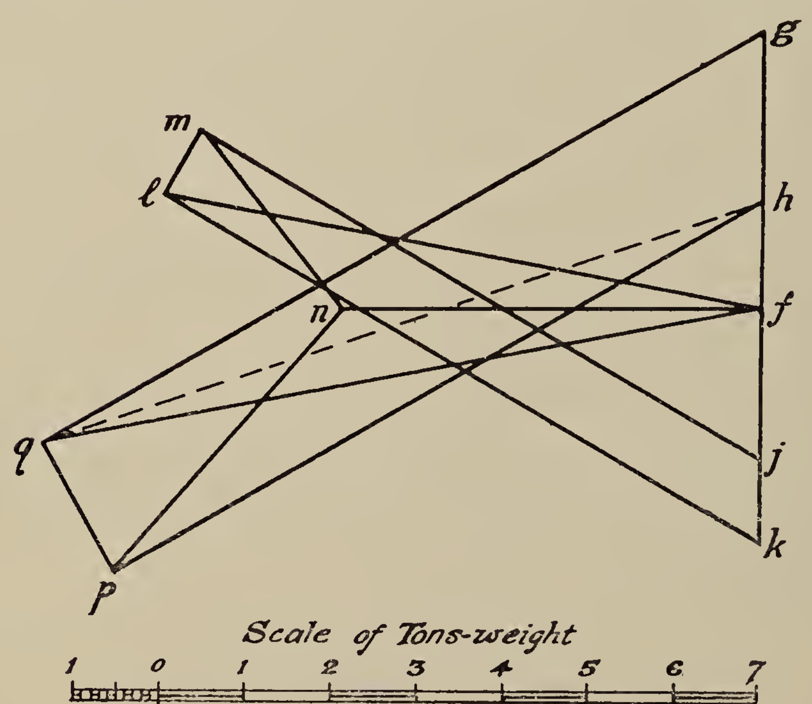


Fig. 92.

We next deal with the forces at the point of intersection of FQ and QP in the same way, obtaining *pn* to represent the force in the bar PN, and *nf* to represent the force in the bar NF.

The remaining points C, D, E, etc., are dealt with in the same way, and our stress diagram is then complete. We measure each line of the stress diagram to obtain the magnitude of the stress in the corresponding member of the roof-truss, and tabulate the results.

To determine whether the stresses in the various members are tensile or compressive we proceed as follows:—Consider the forces acting at any point where two or more bars meet. We name these forces in accordance with the letters which we have given to the spaces each side of them, as already explained: in doing so, however, we must observe the following rule:

The forces acting at any point of intersection are to be known by the letters of the spaces each side of the force *proceeding from space to space in a clockwise direction round the point.*

For example, the forces acting at A are to be known as FG, GQ,

and QF: they must *not* on any account be termed GF, QG, or FQ. Similarly, the forces acting at B are to be known as GH, HP, PQ, and QG, and the forces at C as HJ, JM, MN, NP, and PH.

The external loads are accordingly in this case GH, HJ, and JK, so in commencing our stress diagram we draw *gh* downwards to represent the downward force GH, *hj* downwards to represent the downward force HJ, and *jk* downwards to represent the downward force JK. The upward reactions KF and FG are similarly represented by the upward lines *kf* and *fg*.

We are now in a position to determine whether the stress in a particular bar acts *to* or *from* its ends, i.e., whether the bar is in *compression* or *tension*. Take, for example, the bar joining the points B and C. The stress in this bar produces a force at the point B which we term HP and which is represented on the stress diagram by the line *hp*. From *h* to *p* is *downward and to the left*: therefore the force HP acts downwards and to the left, i.e., towards the point B. The stress in the same bar also produces a force at the point C which we term PH and which is represented on the stress diagram by the line *ph*. From *p* to *h* is *upward and to the right*: therefore the force PH acts upwards and to the right, i.e., towards the point C. We see, therefore, that the stress in this bar produces forces acting outwards, i.e., *pushes* towards the points B and C. It is therefore in *compression*.

Similar reasoning will enable us to determine the nature of the stress in any of the bars: at first the method may seem a little confusing, but after practising it a few times it will be found quite easy. Bars which are in tension are known as *ties*, while those in compression are termed *struts*.

TABLE OF STRESSES.

STRUTS.		TIES.	
Member.	Stress.	Member.	Stress.
QG	7·67 tons-weight	FQ	6·70 tons-weight
QP	1·73 ,,	FN	4·35 ,,
PH	6·67 ,,	NP	2·54 ,,
MJ	6·00 ,,	NM	1·55 ,,
ML	·87 ,,	FL	5·66 ,,
LK	6·50 ,,		

7. A force of 3·7 tons-weight is transmitted by an iron rod $\frac{7}{8}$ inch in diameter. Find the stress intensity.

8. Calculate the intensity of stress in a flat bar, $4\frac{1}{2}$ inches × $\frac{3}{4}$ inch, in which the total stress is 53,000 pounds-weight.

9. The intensity of stress in a steel bar, $1\frac{3}{8}$ inches in diameter, is 7·23 tons-weight per square inch. Find the load carried by the bar.

10. What force in tons-weight can be transmitted by a steel tube, $1\frac{5}{8}$ inches outside diameter and $1\frac{1}{4}$ inches inside diameter, if the stress-intensity is not to exceed 16,000 pounds-weight per square inch ?

11. A bar, 5 inches in width, has to carry a load of 38·9 tons-weight. If the stress is not to exceed $7\frac{1}{2}$ tons-weight per square inch, find the least thickness of the bar.

12. A tie-rod of mild steel has to transmit a tension of 15·84 tons-weight. Determine the diameter required, if the stress is not to exceed 8 tons-weight per square inch.

13. Find the extension produced in a bar, 20 feet in length, by a tension of 7·45 tons-weight per square inch. Take E as 13,000 tons-weight per square inch.

14. How much will a steel bar, 24 feet in length, shorten under the action of a compressive stress of 4·63 tons-weight per square inch, if Young's Modulus for steel is 30,000,000 pounds-weight per square inch ?

15. What is the strain caused in an iron bar, if the stress in it is $5\frac{1}{2}$ tons-weight per square inch, and E for the material of the bar is 26,000,000 pounds-weight per square inch ?

16. An iron tie-rod, $1\frac{1}{8}$ inches in diameter, carries a load of 6·17 tons-weight. Find the stress and strain in the rod, if $E = 12{,}500$ tons-weight per square inch.

17. A steel bar, $2\frac{1}{2}$ inches in width, $\frac{7}{8}$ inch in thickness, and 14 feet in length, carries a tension of 15·8 tons-weight. Determine the stress-intensity, the strain, and the extension. Take E as 13,000 tons-weight per square inch.

18. Find the maximum elastic extension possible in a round rod, $1\frac{1}{4}$ inches in diameter, and 17 feet 6 inches in length, taking E as 29 million pounds-weight per square inch, and the elastic limit as 24 thousand pounds-weight per square inch. If the working stress is not to exceed 50 per cent. of the elastic limit, what load will the rod safely carry ?

19. A strut, 10 feet 6 inches in length, has 8·24 square inches of cross-sectional area, and carries a load of 31·5 tons-weight. The shortening produced is ·037 inch. Determine the stress-intensity, the strain, and the value of Young's Modulus for the material of the strut.

20. Determine the safe load for a solid steel column, 4 inches in diameter, if the working stress is to be 5·85 tons-weight per square inch.

21. What thickness must the metal of a cast-iron column be made, if the external diameter is to be 8 inches, the allowable stress is $2\frac{3}{4}$ tons-weight per square inch, and the load to be carried is 83 tons-weight ?

22. If the Modulus of Rigidity for steel is 12 million pounds-weight per square inch, find the strain produced in a piece of steel by a shear stress of 2·78 tons-weight per square inch.

23. Determine the value of G, if a shearing force of 38 tons-weight, distributed over an area of $13\frac{1}{4}$ square inches, produces a strain of ·000575 radian.

24. Find the value of Young's Modulus for copper, if a force of 4·35 tons-weight, acting upon an area of 1·27 square inches, produces a strain of ·000489.

25. A water-tank, weighing $8\frac{1}{2}$ tons, and containing 5,000 gallons of water, is to be supported on four cast-iron columns, each 5 inches in outside diameter. If the allowable stress in the metal of the columns is 1·53 tons-weight per square inch, what thickness must they be made ? Give the thickness to the nearest eighth of an inch, and take the weight of a gallon of water as ten pounds.

26. The diagram shows a roof-truss of 15 feet span. Draw the frame-diagram and calculate graphically the stresses in the members.

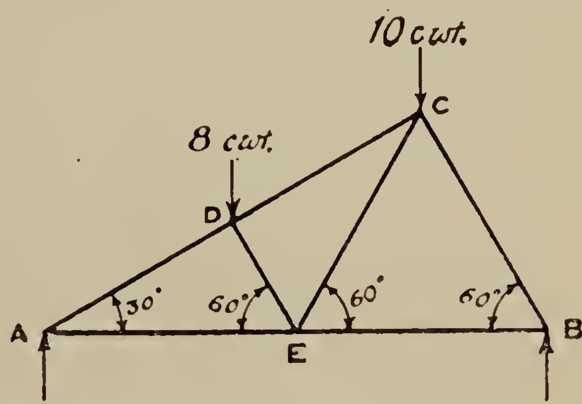


FIG. 93.

27. A frame, ABCDE, is composed of seven rods, AB, BC, CD, EA, EB, EC, and ED, of equal length. BC is horizontal and uppermost. Loads of 5 cwt. are applied at B and C, and each rod weighs 42 pounds. Find the reactions at the supports at A and D, and the stresses in the members.

CHAPTER II: CENTRIPETAL FORCE

WE have already discussed the motion of a body moving along a circular path with uniform speed. We have seen that such a body must have an acceleration, of constant magnitude, directed towards the centre of the circle, this acceleration being necessary in order to change the direction of motion of the body, which would otherwise travel along a straight line. We have found also that the magnitude of this **normal acceleration** is $\mathbf{v^2/r}$, where $\mathbf{v}$ is the linear speed of the body, and $\mathbf{r}$ is the radius of the circle, all the quantities being expressed in systematic units.

We now have to consider circular motion taking into account the **forces** acting upon the body.

If we tie a stone or a chestnut or any other small article to the end of a string, and whirl it round and round, it travels approximately in a circle. As it goes round we feel a pull in the string, as though the stone were trying to get away from us, and if we release our hold of the string, the stone flies off at a tangent. A little thought will show us the reason for these phenomena.

When any body is moving, it is, at any particular instant, travelling in some definite direction, and, unless it is prevented from so doing, it will continue to move in that direction for ever, i.e., it will obey the **First Law of Momentum** and travel in a straight line. In the case of our stone-whirling experiment, since the stone travels in a circle instead of a straight line, this law tells us that the stone must be acted upon by some external force, compelling it to change its direction. Clearly, the **active force** which does this, is the pull exerted by our hand and transmitted, by the tension in the string, to the stone. The pull felt *by* the hand

is merely the **passive force** or **reaction** in the opposite direction.

To the force which acts upon a body and compels it to move in a circle we give the name **centripetal force,** and to the equal and opposite reaction we give the name **centrifugal force.** The centripetal force *alone* acts on the moving body : the centrifugal force merely reacts on the means of constraint.

The essential thing to note, therefore, is that *the real active force is the pull on the stone, which compels it, against its natural tendency, to move in a circle.* We should not think of the stone as trying to get away from our hand, but as trying to continue its natural motion in a straight line, and being compelled by our pull on it to move in a circle.

Clearly the direction of the centripetal force in this case is along the string, that is, it is *towards the centre of the circle* : we shall see that this is always the direction of the centripetal force.

We had already, from a consideration of the **velocity** of a body moving in a circle, seen that it must have an **acceleration** towards the centre of the circle. We have now, from a consideration of the **First Law of Momentum,** seen that there must be a **force** acting on the body.

These two results have been obtained independently, but obviously, when we had found the normal acceleration, we could have applied the laws of momentum, and proceeded to reason at once, that, since the body was subject to an acceleration, it must be undergoing a change of momentum, and that therefore it must be under the action of some external force. As we found that the acceleration was towards the centre of the circle, so we can state that the change of momentum must be also towards the centre of the circle. The **Second Law of Momentum** then tells us at once that the force producing the change of momentum must likewise be directed towards the centre of the circle.

We have then :—

Centripetal force = mass of body $\times$ normal acceleration

$$= M \times \frac{v^2}{r}$$

$$= \frac{M.v^2}{r} \text{ poundals,}$$

where M = mass of body in pounds,
v = linear speed in feet per second,
and r = radius of circular path in feet.

We can also express the same result in terms of ω, the angular velocity of the body in radians per second, for we saw that the normal acceleration could be expressed as $\omega^2 r$ feet per second per second. We have then :—

$$\text{Centripetal force} = M.\omega^2.r \text{ poundals.}$$

If we express the mass of the body in grammes, the speed in centimetres per second, and the radius of the path in centimetres, then the formulæ we have found will give us the value of the centripetal force in dynes.

ROTATION OF LINEAR MOMENTUM. There is yet another way of expressing the magnitude of a centripetal force, for we have :—

$$\begin{aligned}\text{Centripetal force} &= M.\omega^2.r \\ &= M.\omega r.\omega \\ &= M.v.\omega \text{ (since } v = \omega r),\end{aligned}$$

i.e., **Centripetal force = linear momentum $\times$ angular velocity.**

So we may say that the centripetal force required to make a moving body travel in a circle is equal to the product of the linear momentum of the body and the angular velocity with which it travels round the circle.

Sometimes this is put in slightly different words, and we say that the centripetal force *rotates the linear momentum* of the body. When a body travels round a circular path, the *magnitude* of its momentum is unchanged : all that the centripetal force does is to change

the *direction* of the momentum, so that the latter may be said to be rotated about the centre of the circle.

THE CONICAL PENDULUM. Let us think again of the stone tied to the end of a string. If we whirl this round so that the circle in which it moves is horizontal, then we shall have a good example of the arrangement which is termed the **conical pendulum.** If we want to define a conical pendulum, we can say that it consists of a small heavy body tied by a string to a fixed point O, and caused to rotate in a horizontal circle. The name is derived from the fact that the string, as it moves round, describes a cone, of which the axis is vertical and the vertex is at the fixed point O, and from the fact that the motion of the string and body together resembles the motion of a pendulum, as we shall see when we consider periodic motion.

If we actually perform the experiment, we shall find that the angle α, made by the string with the vertical axis, remains constant so long as the speed of the stone remains constant and that the faster we whirl the stone the greater will be the angle α. We shall find, however, that owing to the action of gravity, we cannot possibly make the stone rotate in a horizontal plane through the fixed point O: that is to say, however fast we rotate the stone the angle α is always less than a right angle.

Let us now determine the relations between the mass of the body, the speed of rotation, the angle of inclination of the string, and other quantities concerned. We will assume that the speed is constant.

Let l = the length of the string in feet,
r = the radius of the circular path in feet,
m = the mass of the body in pounds, and
ω = the uniform speed of the body in radians per second.

The **centripetal force** required to cause the circular motion is $m\omega^2 r$ poundals, acting horizontally towards the centre of the circle.

The **weight** of the body is mg poundals acting verti-

cally downwards. This weight and the **tension** of the string, T poundals, are the only external forces acting upon the body. The necessary centripetal force must therefore be provided by the combined effects of these two forces. Now the weight of the body, being vertical, has no horizontal component, so that *the whole of the centripetal force must be due to the horizontal component of the tension in the string.* Also, since the body remains in the same horizontal plane, the vertical forces acting upon it must be in equilibrium amongst themselves; that is, the vertical component of the tension in the string must be equal to the weight of the body in magnitude, and opposite in direction.

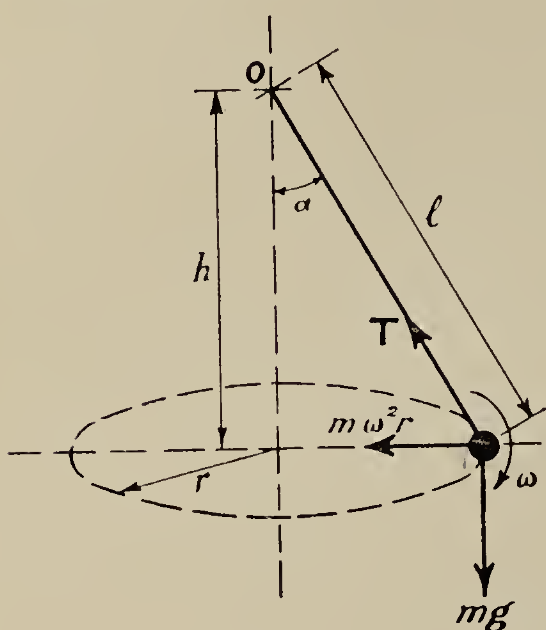


FIG. 94.

We have therefore :—

$$T \cos a = mg \quad . \quad . \quad . \quad (1)$$

and $$T \sin a = m\omega^2 r \quad . \quad . \quad . \quad (2)$$

Dividing each side of (1) by the corresponding side of (2), we get :—

$$\cot a = \frac{g}{\omega^2 r} \quad . \quad . \quad . \quad (3)$$

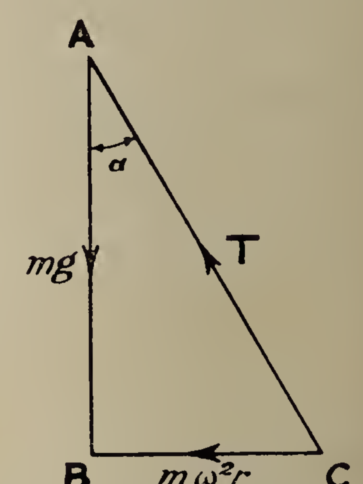


FIG. 95.

Now $\cot a = h/r$, where h is the vertical height of the fixed point, O, above the plane of rotation. Therefore we have :—

$$\frac{h}{r} = \frac{g}{\omega^2 r},$$

whence $$h = \frac{g}{\omega^2} \text{ feet} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which shows us that the **vertical height, h,** is independent of the length of the string or the mass of the

body, and depends *only* on the speed of rotation and the value of g, which latter may be considered constant. This is an interesting and rather surprising result, for one would hardly expect to find that neither altering the length of the string, nor changing the mass of the body would affect the vertical height h.

It might be asked, what would happen if the length of the string were actually altered? The answer is that the body would simply move nearer to the axis of rotation if the string were shortened, or further from the axis if the string were lengthened, until, in either case, the vertical height was the same as before (see Fig. 96).

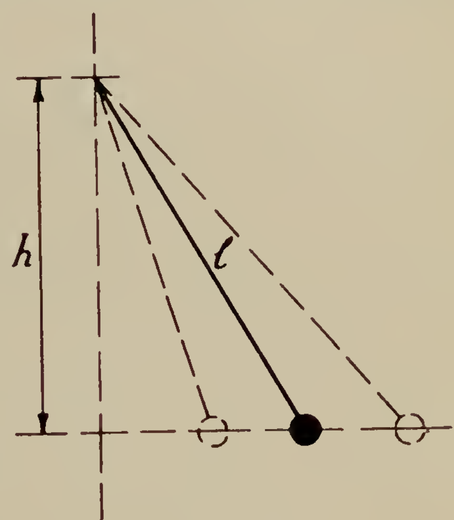


FIG. 96.

If required, we can, of course, express the height h in terms of the number of revolutions per minute, instead of the number of radians per second. We have :—

N revolutions per minute

$$= \frac{N}{60} \text{ revolutions per second}$$

$$= \frac{2\pi N}{60} \text{ radians per second}$$

$$= \frac{\pi N}{30} \text{ radians per second.}$$

Therefore $\omega = \pi N/30$, and substituting this value in equation (4) above, we have :—

$$h = \frac{g}{\left(\frac{\pi N}{30}\right)^2}$$

$$= \frac{900.g}{\pi^2.N^2} \text{ feet} \quad . \quad . \quad . \quad . \quad (5)$$

The **tension** in the string is given by :—

$$\begin{aligned} T &= mg.\sec\,\alpha \\ &= mg.l/h \\ &= mgl \times \omega^2/g \\ &= ml\omega^2 \text{ poundals} \quad . \quad . \quad . \quad . \quad (6) \end{aligned}$$

The **time** of one complete revolution is given by :—

$$\begin{aligned} t &= \frac{\text{angle turned through in one revolution}}{\text{angular velocity}} \\ &= \frac{2\pi \text{ radians}}{\omega \text{ radians per second}} \\ &= \frac{2\pi}{\omega} \text{ seconds} \\ &= 2\pi.\sqrt{\frac{h}{g}} \text{ seconds} \quad . \quad . \quad . \quad . \quad (7) \end{aligned}$$

since $h = g/\omega^2$ and therefore $\omega = \sqrt{g/h}$.

Here, again, the result is independent both of the mass of the body and of the length of the string : this we should expect, since the time of revolution must obviously depend upon the speed, and if that be constant, can be affected by nothing else.

The conical pendulum is employed in engineering, in the form of the **simple governor,** for controlling the supply of steam to steam engines.

OVERTURNING OF VEHICLES ON CURVES. A vehicle in motion, like any other moving body, will continue to travel in a straight line unless it is compelled by the action of external force to do otherwise. If it is to move in a circular path, then it must be acted upon by centripetal force, in accordance with the laws of momentum.

Let us consider how this centripetal force is provided. In the case of a road vehicle, such as a motor-car, the centripetal force is produced by the lateral friction between the road surface and the tyres of the vehicle. In the case of such a vehicle as a tram-car, it is produced by the side pressure of the rails on the flanges

of the wheels. In either case it acts at the lowest part of the vehicle, and therefore much below the centre of mass.

Now if a body is to move in a circle, *every particle* of it must be acted upon by a centripetal force proportional to the mass of the particle, and the resultant of all these little forces will be the centripetal force necessary to make the whole body move in a circle. But the resultant of such a system of forces acting on the particles must be a force acting through the centre of mass of the body. Therefore *the required centripetal force must act through the centre of mass of the body.*

Now, in the case of many vehicles, the only available force is, as we have seen, applied at the lowest part of the wheels, instead of at the centre of mass where we require it. However, we can utilise this force all the same, for we proved in Chapter 3 (page 54), that *a single force, acting at any point, is equivalent to the same force acting at any other point, plus a couple of which the moment is equal to the product of the force and the perpendicular distance between its two positions.*

We see, therefore, that the lateral force applied to the wheels of a vehicle is equivalent to a parallel force acting at the centre of mass of the vehicle, plus a couple whose moment is equal to the product of the force and the vertical height of the centre of mass above the road (or rails). The *transferred force* will make the vehicle move in a circle: the *couple* will tend to overturn the vehicle sideways, away from the centre of the circle.

The couple produced in this way is termed the **overturning couple,** and its moment the **overturning moment.**

Therefore the actual lateral force applied at the wheels has two effects. It produces (*a*) the **centripetal force** acting at the centre of mass and making the vehicle travel in a circle, and (*b*) the **overturning couple** tending to upset the vehicle sideways. In Fig. 97, P_1 is the actual force applied to the wheels, P_2 and P_3 the added forces, which, being equal and

opposite, balance each other. P_3 is then the centripetal force, while P_1 and P_2 constitute the overturning couple.

In every such case, therefore, the mere fact of the vehicle moving in a curved path must produce a tendency to overturn. Since the centripetal force is directly proportional to the square of the speed, and inversely proportional to the radius of the circle, so also the overturning moment must be directly proportional to the square of the speed, and inversely proportional to the radius. The tendency to overturn will therefore increase rapidly as the speed increases, and will be greater as the radius of the path is smaller. Clearly, if a vehicle attempts to take a corner sharply and at high speed it will be likely to come to grief.

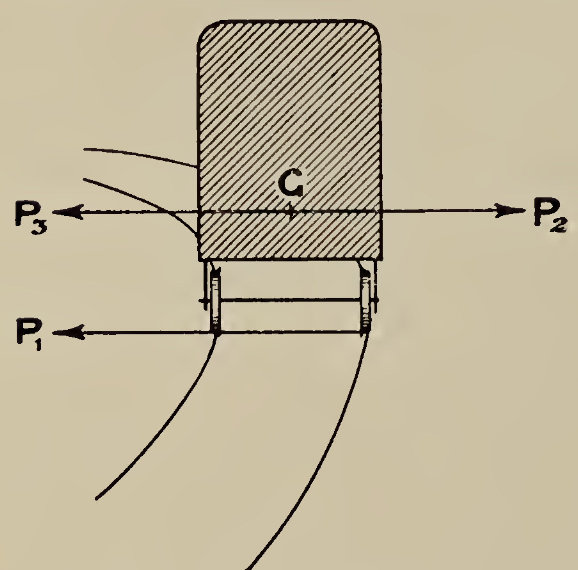


Fig. 97.

The overturning couple is, however, opposed by another couple, formed by the weight of the vehicle acting vertically downwards, and the reaction of the ground acting vertically upwards on the wheels. This couple tends to prevent the vehicle from overturning, and is therefore known as the **righting couple**, and its moment as the **righting moment.** Plainly the condition of equilibrium, as regards upsetting, is that the righting moment must equal the overturning moment.

As we shall see shortly, the righting moment increases step by step with the overturning moment, the one exactly balancing the other, up to a certain value which is the maximum possible for the *righting* moment. But the *overturning* moment can go on increasing indefinitely, as the speed increases. Therefore, for any given conditions, we can determine the maximum speed which is consistent with lateral stability.

Consider the forces acting upon a vehicle when it is moving in a circle, as indicated in Fig. 98.

W is the weight of the vehicle, acting vertically downwards through the centre of gravity, G. N_1 and N_2 are the vertical reactions of the ground, acting upwards on the wheels: they are together equal to the weight W.

P_1 is the force of friction (lateral) between the wheel tyres and the ground (or the thrust of the rails on the flanges of the wheels, as the case may be): P_2 and P_3 are the equal and opposite forces introduced at G, so that $P_1 = P_2 = P_3$.

P_3 is then the **centripetal force,** compelling the vehicle to move in a circle, and acting towards the centre of the circle. P_1 and P_2 together constitute the **overturning couple.**

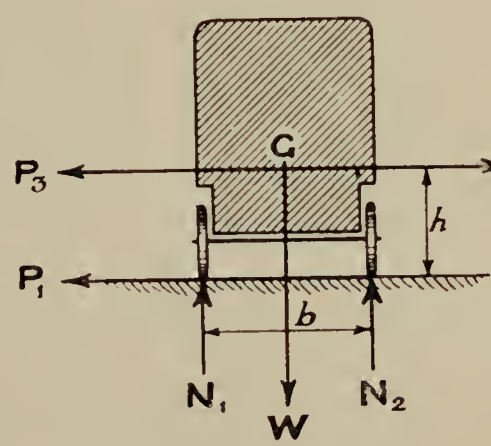


FIG. 98.

We may here remind ourselves that the *centrifugal* force is merely the reaction *by* the wheels on the road or rails. It is *not* a force acting on the vehicle, and we need not therefore take it into account.

Let h be the vertical height of the centre of gravity, G, of the vehicle, above the ground, and b the lateral distance between the wheels.

The moment of the overturning couple is then $P_2.h$.

The moment of the righting couple is $(N_2.\frac{1}{2}.b - N_1.\frac{1}{2}.b)$, i.e., $\frac{1}{2}(N_2 - N_1)b$. This will be evident if we take moments about any point on the line of action of the weight W.

If the vehicle is going straight ahead, then $P_1 = P_2 = P_3 = O$, so that the overturning moment $= O$. Also N_1 must then be equal to N_2, that is, the weight of the vehicle must be equally distributed between the near and off wheels, and the righting moment $= O$.

If the vehicle is moving in a circle, the greatest

value that the *righting* moment can have is when $N_1 = O$ and $N_2 = W$, that is, when the whole weight of the vehicle is borne by the outer wheels, and the inner wheels are on the point of leaving the ground. (Obviously N_1 can never be negative, that is, the ground can never exert a *pull* downwards on the wheels.)

Clearly this will be equal to the greatest value which the *overturning* moment can have without the vehicle commencing to upset. We then have :—

Overturning moment = righting moment

or
$$P_2.h = \tfrac{1}{2}(W - O)b$$
$$= \tfrac{1}{2}.W.b.$$

This gives us a maximum value of the centripetal force, to avoid overturning, of :

$$P_3 = P_2 = \frac{W.b}{2.h} \quad . \quad . \quad . \quad . \quad (1)$$

If P_3 exceeds this value, then the vehicle will commence to overturn. Let M = mass of vehicle, and v = its linear speed. Then we have :—

$$W = \text{weight of vehicle} = M.g$$

and
$$P_3 = \text{centripetal force} = \frac{M.v^2}{r}.$$

Substituting these values for P_3 and W in (1) above, we have :—

$$\frac{M.v^2}{r} = \frac{M.g.b}{2h},$$

whence
$$v^2 = \frac{g.b.r}{2.h}$$

and
$$v = \sqrt{\frac{g.b.r}{2.h}} \quad . \quad . \quad . \quad (2)$$

This gives the maximum linear speed at which the vehicle can travel round a circular path without commencing to overturn. If b, g, r, and h are in F.P.S. units, then v will be the speed in feet per second : if

they are in C.G.S. units then v will be in centimetres per second.

As we should naturally expect, the allowable speed v is *increased* if the breadth b of the wheel-base is increased, or if the radius r of the circle is increased: it is *decreased* if the height h of the centre of mass of the vehicle above the ground is increased.

CHECK BY DIMENSIONS. The dimensions of the right-hand side of equation (2) above are :—

$$\left\{\frac{[L][T]^{-2} \times [L] \times [L]}{[L]}\right\}^{\frac{1}{2}}$$

$$= \{[L]^2[T]^{-2}\}^{\frac{1}{2}} = [L][T]^{-1}$$

which are the correct dimensions for a linear speed.

UTILISATION OF GRAVITY TO PRODUCE CENTRIPETAL FORCE. The method we have considered above, of obtaining the necessary centripetal force by acting on the wheels of the vehicle at their lowest part, is open to several objections. These objections apply particularly to vehicles travelling at high speed, such as railway trains, and motor vehicles on a racing track. The disadvantages are due to (i) frictional effects, and (ii) overturning effects.

In the case of a tramway, for example, where the requisite centripetal force is produced by the pressure of the rails against the flanges of the wheels, the effects of friction include :—

(i) wear and tear of rails,
(ii) wear and tear of wheel flanges,
and (iii) retarding effect on speed, due to increase of tractive resistance.

The overturning effects we have already considered.

All these consequences are extremely undesirable on railways, where, owing to the relatively high speed, they would be most conspicuous.

The remedy adopted is to produce the necessary centripetal force in an entirely different way, namely, by utilising the force of gravity. By this means, as

we shall see, it is possible to do away entirely with both the lateral friction effects and the tendency to overturn: at least within the limits of unavoidable practical error.

The principle is applied by tilting up the track so that the rail which is farther from the centre of the circle is raised above the inner rail. The weight, W, of the vehicle acts, as always, vertically downwards, but the resultant reaction, R, from the rails on to the wheels, being perpendicular to the track, is no longer vertical: therefore W and R are no longer in line and no longer balance each other. If the angle of slope of the track is suitably chosen, the resultant, P, of W and R will just provide the horizontal force necessary to make the vehicle move in a circle, that is, the centripetal force will be provided by the combined action of the weight of the vehicle and the reaction of the rails on the vehicle, as shown in Fig. 99.

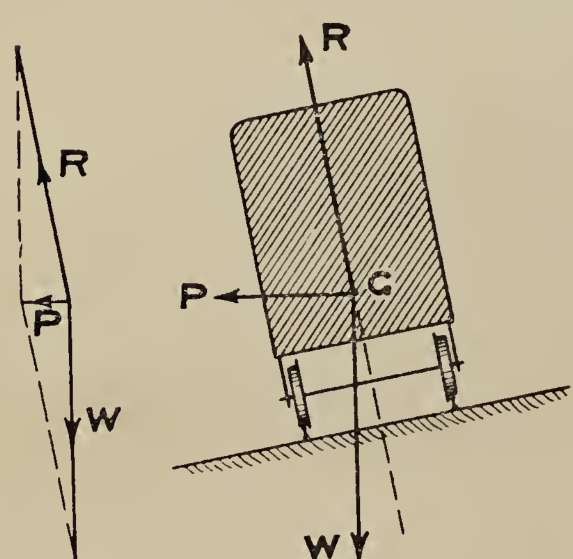


Fig. 99.

In the case of a railway, the amount by which the outer rail is raised above the inner rail is known as the **superelevation** of the outer rail, or sometimes as the **cant.** This superelevation is only a matter of a few inches, for if the speed of the trains is high, the radius of the curves on a railway is considerable also.

In the case of racing tracks for motor-cars, the **angle of cant** may be considerable, as high speeds may be combined with relatively small radius of curvature.

DETERMINATION OF SUPERELEVATION REQUIRED.

Let v = speed of train in feet per second,
r = radius of circle in feet,
b = distance between rail centres in feet,
and θ = angle of cant.

Then, if there is no side thrust on the rails, we have, on resolving parallel to the track slope :—

$$W \sin \theta = P \cos \theta,$$

whence $$\frac{\sin \theta}{\cos \theta} = \frac{P}{W},$$

so that $$\tan \theta = \frac{Mv^2/r}{Mg}$$ (since P is the centripetal force and W the weight of the vehicle)

$$= \frac{v^2}{rg}$$

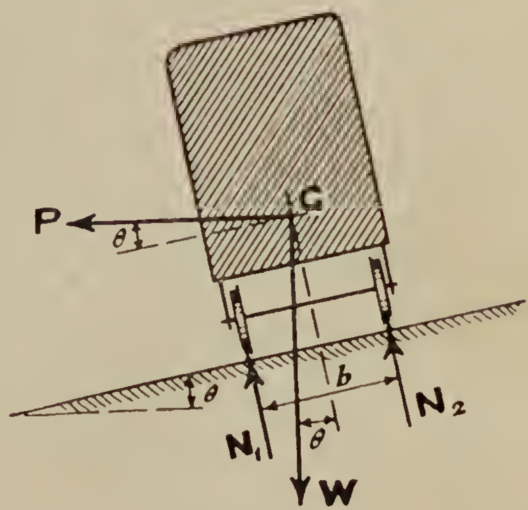


Fig. 100.

This enables us to determine the angle of slope of the track: to find the actual distance which the outer rail must be raised above the inner we proceed as follows :—

Amount of superelevation required

$$= s = b.\sin \theta$$

$$= b.\tan \theta$$ approximately, provided that θ is small, as it will be on a railway

$$= \frac{bv^2}{rg} \text{ feet}$$

$$= \frac{12.bv^2}{rg} \text{ inches.}$$

If b, v, r, and g are all expressed in C.G.S. units, then the superelevation required will be

$$s = \frac{bv^2}{rg} \text{ centimetres.}$$

In the case of a motor track, the angle of slope, θ, is all that is required, and this can be obtained from the value of $\tan \theta$.

OTHER CONSIDERATIONS AFFECTING CANT. Two other considerations affecting the amount of cant required may be mentioned here. The first is

gyroscopic action on the wheels of the vehicle : this will be dealt with in a later chapter, so that all weneed say about it, now, is that it produces an overturning effect in the *same* direction as that produced by centripetal action.

The second consideration only affects *trains* of vehicles. In any such case, three forces act on each vehicle (except the engine and the tail coach) and have components in the direction of motion. These three forces are (*a*) the pull from the vehicle in front, (*b*) the tractive resistance of the vehicle itself, and (*c*) the pull from the vehicle behind. These will all be in different directions (see Fig. 101), each of them being tangential to the curve at a different point. Consequently they will have a resultant, of which the direction will be approximately towards the centre of the circle.

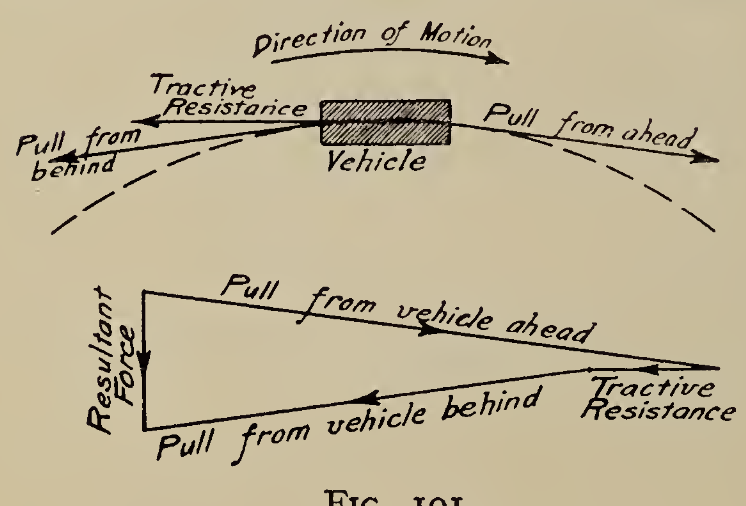


Fig. 101.

This resultant force, due to the **tension along the train,** will not be the same for each vehicle, but it will, in each case, supply some part of the requisite centripetal force, so that its general effect will be to *reduce* the amount of cant needed.

A case is mentioned in Whitelaw's "Surveying," where, on the Cheshire Lines railway, the 4-inch cant put in from calculation was found, in practice, to be excessive : experience showed that $1\frac{3}{4}$ inches was adequate. Probably the discrepancy was due, in part at least, to the tension along the train.

SUMMARY OF CHAPTER 11

The force required to change the *direction* of motion of a body, and to compel it to move in a curved path, is termed the **centripetal force.** The equal and opposite reaction on the means of constraint is known as the **centrifugal force.**

When a body moves in a circle, we have :—

Centripetal force = mass of body × normal acceleration

$$= M\frac{v^2}{r} \text{ poundals} = M.\omega^2.r \text{ poundals} = M.v.\omega \text{ poundals}$$

$$= \text{linear momentum} \times \text{angular velocity.}$$

The **conical pendulum** consists of a small heavy body tied by a string to a fixed point and caused to rotate in a horizontal circle.

The **height** of the conical pendulum $= \frac{g}{\omega^2}$ feet.

The **tension** in the string $= ml\omega^2$ poundals.

The **time** of one revolution $= 2\pi\sqrt{\frac{h}{g}}$ seconds.

The conical pendulum is employed in engineering as the **simple governor** for steam engines.

If, in order to produce circular motion, a force is applied at the lowest point of a vehicle, this force is equivalent to a **centripetal force** acting at the centre of mass, plus an **overturning couple** tending to upset the vehicle. This tendency is resisted by the **righting couple** formed by the weight of the vehicle, and the reactions of the ground.

The maximum speed at which a vehicle can travel round a circle of radius r, without commencing to overturn, is given by :—

$$v = \sqrt{\frac{bgr}{2h}},$$

where b is the distance between the wheels, laterally, and h is the height of the centre of gravity of the vehicle above the ground.

To avoid frictional effects and overturning tendencies, the necessary centripetal force on a railway is provided by raising the outer rail above the level of the inner rail, thereby utilising the force of gravity.

The **angle of cant** is given by :— $\tan\theta = \frac{v^2}{rg}$.

The amount of **superelevation** required is $\frac{bv^2}{rg}$ (systematic units).

Other considerations affecting the amount of cant required are **gyroscopic action** and the **tension along a train.**

EXAMPLES XI

(*For Hints on Working Examples, see page* 21.)

1. *A body of* 28 *pounds mass is moving in a circle of* 13 *feet radius, with a linear speed of* 44 *feet per second. Find the centripetal force*

acting upon it. Find also the centripetal force (a) if the mass were doubled, (b) if the speed were doubled, and (c) if the radius were doubled.

Centripetal force

$$= \frac{\text{mass of body} \times \text{square of speed}}{\text{radius of circle}}$$

$$= \frac{28 \text{ pounds} \times (44 \text{ feet per second})^2}{13 \text{ feet}}$$

$$= \frac{28 \times 44 \times 44}{13} \text{ poundals}$$

$$= 4{,}170 \text{ poundals}$$

$$= 129\tfrac{1}{2} \text{ pounds-weight.}$$

If the *mass* were doubled, then the centripetal force would be doubled also, and would therefore be

$$= 8{,}340 \text{ poundals}$$

$$= 259 \text{ pounds-weight.}$$

If the *speed* were doubled, then the centripetal force would be four times as great, and would therefore be

$$= 16{,}680 \text{ poundals}$$

$$= 518 \text{ pounds-weight.}$$

If the *radius* were doubled, then the centripetal force would be halved, and would therefore be

$$= 2{,}085 \text{ poundals}$$

$$= 64\tfrac{3}{4} \text{ pounds-weight.}$$

2. *A stone, of $\frac{1}{2}$ pound mass, is whirled round in a vertical circle on the end of a piece of string* 2 *feet in length, making* 40 *revolutions per minute. Find the tension in the string (a) at the top of the circle, and (b) at the bottom of the circle. If the string is released when the stone is at the top of the circle, with what velocity will the stone move away? Assume constant speed of rotation.*

Centripetal force on stone

$$= (\text{mass of stone}) \times (\text{square of angular velocity}) \times (\text{radius of circle})$$

$$= \tfrac{1}{2} \text{ pound} \times \left(\frac{40}{60} . 2\pi \text{ radians per second}\right)^2 \times 2 \text{ feet}$$

$$= \tfrac{1}{2} \times \left(\frac{4}{3}\pi\right)^2 \times 2 \text{ poundals}$$

$$= 16\pi^2/9$$

$$= 17{\cdot}55 \text{ poundals.}$$

Tension in string at top of circle

$$= \text{centripetal force} - \text{weight of stone}$$
$$= 17{\cdot}55 \text{ poundals} - 16{\cdot}1 \text{ poundals}$$
$$= 1{\cdot}45 \text{ poundals.}$$

Tension in string at bottom of circle

$$= \text{centripetal force} + \text{weight of stone}$$
$$= 17{\cdot}55 \text{ poundals} + 16{\cdot}1 \text{ poundals}$$
$$= 33{\cdot}65 \text{ poundals.}$$

If the string is released when the stone is at the top of the circle, the stone will move away with the velocity which it possesses at that instant, i.e., it will move off in a horizontal direction with velocity of magnitude v such that :—

$$v = \text{angular velocity in radians per second} \times \text{radius of circle in feet}$$
$$= \frac{4}{3}\pi \times 2 \text{ feet}$$
$$= 8{\cdot}38 \text{ feet per second.}$$

3. *How fast must a bucket of water be whirled in a vertical circle of 2 feet 6 inches radius so that no water may be spilled?*

Let M = mass of water in pounds,
ω = angular velocity in radians per second,
r = radius of circle in feet.

Centripetal force required to make bucket move in circle

$$= \mathrm{M}.\omega^2.r \text{ poundals.}$$

Weight of water

$$= \mathrm{M}.g \text{ poundals.}$$

If the weight exceeds the centripetal force required, then the water will be spilled: therefore the centripetal force must at least be equal to the weight of the water, i.e.,

$$\mathrm{M}.\omega^2.r = \mathrm{M}.g,$$

whence $\omega^2.r = g$

and

$$\omega = \sqrt{g/r}$$
$$= \sqrt{32{\cdot}2/2{\cdot}5}$$
$$= 3{\cdot}59 \text{ radians per second}$$
$$= 3{\cdot}59 \times 60/2\pi$$
$$= 34{\cdot}3 \text{ revolutions per minute.}$$

4. *A simple governor, of which the arms are* 12 *inches in length, rotates at a speed of* 60 *revolutions per minute. Find the vertical height. What percentage increase or decrease of speed is necessary in order that the angle made by the arms with the spindle may be* 45 *degrees?*

A simple governor is merely a conical pendulum adapted to control the speed of an engine.

Let M = mass of ball in pounds,
T = tension in arm in poundals,
ω = angular velocity in radians per second,
l = length of arm in feet,
h = vertical height in feet,
r = radius of rotation in feet,
α = angle made by arm with spindle.

Resolving vertically :—

$$T.\cos\alpha = Mg.$$

Resolving horizontally :—

$$T.\sin\alpha = M\omega^2.r,$$

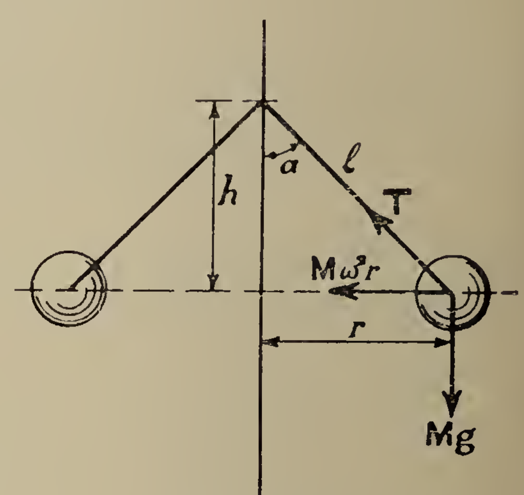


FIG. 102.

therefore
$$\frac{T.\cos\alpha}{T.\sin\alpha} = \frac{Mg}{M\omega^2.r},$$

whence
$$\cot\alpha = \frac{g}{\omega^2.r}$$

and
$$r.\cot\alpha = \frac{g}{\omega^2},$$

i.e.,
$$h = \frac{g}{\omega^2}$$

$$= \frac{32{\cdot}2}{4.\pi^2} \quad \text{(for } \omega. = 60 \text{ r. p. min.} = 1 \text{ rev. per sec.} = 2\pi \text{ rads. per sec.)}$$

$$= \frac{32{\cdot}2}{4 \times 9{\cdot}87}$$

$$= {\cdot}816 \text{ foot}$$

$$= 9{\cdot}8 \text{ inches.}$$

Also from above we have :—

$$\omega^2 = \frac{g}{h}.$$

When $\alpha = 45°$, then $h = l.\cos 45°$
$= 1 \text{ foot} \times {\cdot}707$
$= {\cdot}707 \text{ foot},$

so that $\omega^2 = \frac{32 \cdot 2}{\cdot 707}$

$= 45 \cdot 55$

and $\omega = 6 \cdot 75$ radians per second

$= \frac{6 \cdot 75 \times 60}{2\pi}$

$= 64 \cdot 5$ revolutions per minute.

Therefore increase of speed required

$$= \frac{64 \cdot 5 - 60}{60} \times 100$$

$$= \frac{450}{60}$$

$= 7\frac{1}{2}$ per cent. increase.

5. *What superelevation must be given to the outer rail of a standard gauge* (4 *feet* $8\frac{1}{2}$ *inches*) *railway, on a curve of* $\frac{1}{2}$*-mile radius, in order that a train may travel along it at* 50 *miles per hour without side-thrust between rails and wheel-flanges? If a train passes round the curve at* 30 *miles per hour, what will be the side-thrust per ton mass of train?*

Let P = centripetal force required,
W = weight of vehicle,
α = angle of cant necessary.

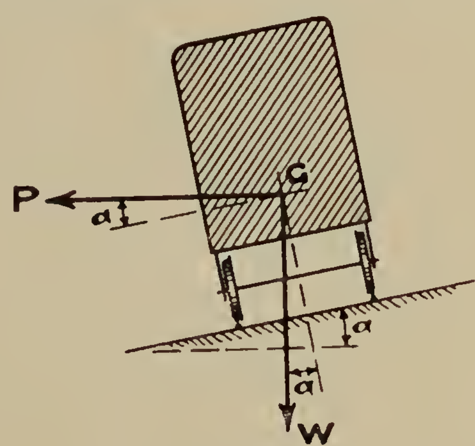


FIG. 103.

Resolving parallel to slope of track

$W.\sin \alpha = P.\cos \alpha,$

whence $\frac{\sin \alpha}{\cos \alpha} = \frac{P}{W}$

$= \frac{Mv^2}{r.Mg}$ (where M = mass of vehicle in pounds, v = speed in ft. per sec., and r = radius of curve in feet),

i.e., $\tan \alpha = \frac{v^2}{rg}$

$= \frac{220 \times 220}{3 \times 3 \times 2{,}640 \times 32 \cdot 2}$

(since 50 m.p.h. $= \frac{220}{3}$ f.p.s.)

$= \cdot 06325.$

Superelevation required

$$= \text{gauge} \times \sin \alpha$$
$$= \text{gauge} \times \tan \alpha \text{ approx. (since } \alpha \text{ is small)}$$
$$= 56\tfrac{1}{2} \text{ inches} \times \cdot 06325$$
$$= 3 \cdot 57 \text{ inches}$$

Side-thrust at speed of 30 miles per hour

$$= \text{W}.\sin \alpha - \text{P}.\cos \alpha$$
$$= \text{M}g.\sin \alpha - \text{M}\frac{v^2}{r} \cos \alpha$$
$$= \text{M}.(32\cdot 2 \times \cdot 06325) - \text{M}\left(\frac{44 \times 44}{2{,}640} \times \cdot 9980\right)$$
$$\text{(since 30 m.p.h.} = 44 \text{ f.p.s. and } \alpha = 3^\circ\ 37')$$
$$= \text{M}(2\cdot 037 - \cdot 732) \text{ poundals}$$
$$= 1\cdot 305.\text{M poundals.}$$

To obtain side-thrust *per ton* we must make M = one ton = 2,240 pounds.

∴ Side-thrust per ton mass of train

$$= (1\cdot 305 \times 2{,}240) \text{ poundals}$$
$$= \frac{1\cdot 305 \times 2{,}240}{32\cdot 2} \text{ pounds-weight}$$
$$= 90\cdot 8 \text{ pounds-weight.}$$

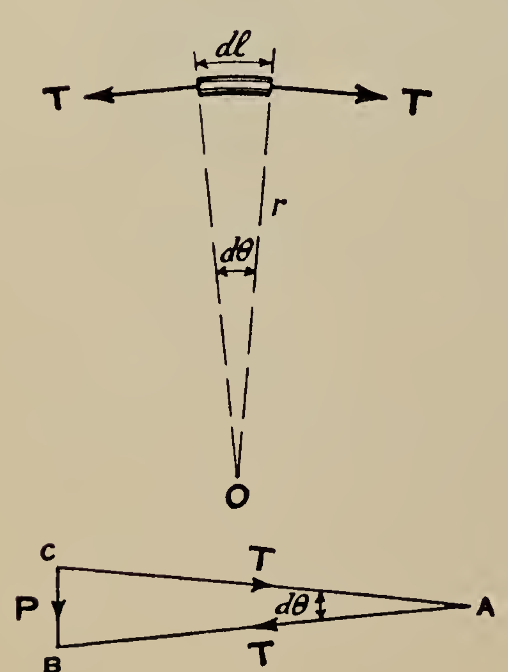


Fig. 104.

6. *Prove that the normal acceleration of a point which is describing a circle of radius r with velocity v is v^2/r.*

A uniform circular wire of small cross-section is rotating in its own plane about its centre with uniform velocity v; by consideration of the forces on a small portion of the wire, or otherwise, prove that the resultant action across a section of the wire is a tension mv^2/g pounds-weight, m being the mass of the wire in pounds per unit length.

If the wire weighs 490 *pounds per cubic-foot and can just stand a stress of* 90,000 *pounds-weight per square inch, show that the greatest value of v is about* 920 *feet per second.* (Univ. Lond. Int. Eng. 1914.)

(*a*) See Chapter 7.

(*b*) Consider the forces acting on an indefinitely small portion of the wire of length *dl* (Fig. 104). These forces are the tensions T acting

at each end of the portion, tangential to the wire. Their resultant gives the required centripetal force P acting normal to the wire.

Let $d\theta$ be the angle subtended at the centre of the circle by the small portion of the wire under consideration.

Drawing the abbreviated parallelogram of forces ABC, we see that the angle BAC $= d\theta$, so that

$$T = \frac{P}{d\theta} \quad (d\theta \text{ being indefinitely small}).$$

As the length of the portion of wire is dl, $d\theta = dl/r$, where r is the radius of the circle. Also $P = m.dl.v^2/r$ poundals. Therefore

$$\begin{aligned} T &= \frac{m.dl.v^2}{r} \div \frac{dl}{r} \\ &= mv^2 \text{ poundals} \\ &= \frac{mv^2}{g} \text{ pounds-weight.} \qquad (Q.E.D.) \end{aligned}$$

(*c*) We have
$$m = \frac{490}{144} \times \frac{\pi d^2}{4} \text{ pounds per foot length}$$

and
$$T = 90{,}000 \times \frac{\pi d^2}{4} \text{ pounds-weight,}$$

therefore
$$90{,}000 \times \frac{\pi d^2}{4} = \frac{490}{144} \times \frac{\pi d^2}{4} \times \frac{v^2}{g},$$

whence
$$90{,}000 = \frac{490}{144} \times \frac{v^2}{32{\cdot}2},$$

so that
$$\begin{aligned} v^2 &= \frac{90{,}000 \times 32{\cdot}2 \times 144}{490} \\ &= 852{,}000 \end{aligned}$$

and
$$v = 923 \text{ feet per second}$$

(or taking g as 32 feet per second per second instead of 32·2 feet per second per second)

$$v^2 = 846{,}000$$

and
$$v = 920 \text{ feet per second.} \qquad (Q.E.D.)$$

7. Find the centripetal force necessary to make a body of 240 pounds mass, moving with a linear speed of 25 miles per hour, travel in a circle of 100 yards radius.

8. A locomotive of 84 tons mass travels round a curve of 550 yards radius at a speed of 35 miles per hour. Find its normal acceleration and the centripetal force required to produce that acceleration.

9. A force of 966 poundals acts upon a body which is moving at a speed of 1,000 yards per minute, and compels it to travel in a circle of a quarter of a mile radius. What is the mass of the body? The force always acts at right angles to the path.

10. A body, of 45 kilogrammes mass, moves along a circular path of 13 feet 6 inches diameter, under the action of a force of a thousand million dynes, which is directed towards the centre of the circle. Find the speed of the body.

11. A body of 44 ounces mass travels in a circle of $17\frac{1}{2}$ inches diameter, making 2,300 revolutions per minute. Find the centripetal force acting upon the body.

12. Find the radius of the circular path of a body of 34 pounds mass, if it travels 5 times round the path in 2·2 seconds under the action of a centripetal force of 25 pounds-weight.

13. A piece of brass of 7 ounces mass is tied to the end of a piece of string 18 inches in length and whirled round in a vertical circle, at a uniform speed of 50 revolutions per minute. Determine the tension in the string (i) when the piece of brass is at its highest point, and (ii) when it is at its lowest point. What is the lowest uniform speed at which the motion could be maintained ?

14. A simple governor is intended to run at a mean speed of 80 revolutions per minute. What length must the arms be made so that their angle of inclination may be 45 degrees ?

15. A simple governor rotates at a speed of 100 revolutions per minute, and the length of the arms is $3\frac{3}{4}$ inches. Find the angle of inclination of the arms from the vertical axis of rotation. What will be the variation of vertical height of the governor for an increase or decrease of speed of 5 per cent. ?

16. Determine the amount of superelevation which must be given to the outer rail of a railway, of 5 feet 6 inches gauge, on a curve of $\frac{1}{4}$-mile radius, so that a train may travel along it at a speed of 35 miles per hour without side-thrust between rails and wheel-flanges. Neglect gyroscopic action and the tension along the train.

17. A curve of 500 metres radius, on a metre-gauge railway, is given a superelevation of the outer rail of 4 centimetres. For what speed, in kilometres per hour, will all side-thrust on the wheel flanges be avoided ?

18. Find at what angle of slope a motor racing track must be laid, on a curve of 200 yards radius, so that a car may travel round it at a speed of 75 miles per hour without lateral friction between the tyres and the track surface.

19. If the amount of superelevation given to the outer rail on a line of railway of 4 feet $8\frac{1}{2}$ inches gauge is not to exceed 5 inches, what is the fastest speed at which a train can travel along the line round a curve of 30 chains radius, without causing pressure between the rails and the flanges of the wheels ? 80 chains are equal to one mile.

20. A horizontal circular disc, 5 feet in diameter, is made to rotate at a uniform speed of 25 revolutions per minute. A block of wood rests on the disc and rotates with it. If the coefficient of friction between the disc and the block is ·35, find how far from the centre of the disc the block can be placed so that it will remain on the disc.

21. In the previous question, if the block is placed at a distance of 2 feet 5 inches from the centre of the disc, what is the highest speed at which the disc can be rotated so that the block will not fly off ?

22. A mass of 56 pounds is fixed to the end of a steel rod, $\frac{1}{4}$ inch in diameter, and 4 feet in length. If the breaking stress for steel is 30 tons-weight per square inch, and the mass is caused to rotate about the other end of the rod, at what speed in revolutions per minute will the rod break ?

23. A body of 5 pounds mass is tied to one end of a string, of which the other end is attached to a fixed point. The body is held so that the string is taut and horizontal, and is then released. Find the tension in the string (*a*) when it is at an angle of 60 degrees from the horizontal, and (*b*) when it is vertical.

24. A conical pendulum rotates at a speed of 72 revolutions per minute, and makes an angle of 50 degrees with the axis of rotation. If the mass of the bob is $2\frac{1}{4}$ pounds, find the tension in the string.

25. The length of a conical pendulum is 16 inches : how fast must it rotate in order that the angle between the string and the axis of rotation may be 85 degrees ? What is then the tension in the string, if the mass of the bob is 14 ounces ?

26. A motor vehicle travels round a curve of 15 yards radius on level ground at a speed of 10 miles per hour. If the mass of the vehicle is 2·35 tons, the breadth of the wheel base is 5 feet, and the height of the centre of gravity above the ground is 3 feet 4 inches, what are the moments of the overturning couple and the righting couple ? What is the greatest speed at which the vehicle could take the curve without commencing to overturn ?

27. A car is running at a uniform speed of 21 miles per hour on level ground. If the height of the centre of gravity is 2 feet 9 inches above the ground, and the wheel base is 4 feet 9 inches in breadth, what is the smallest radius curve which the car can take without slowing down, and without commencing to overturn ?

CHAPTER 12: WORK AND ENERGY

THERE is little need to explain what is meant by **work.** We are, most of us, familiar with it in everyday affairs, and in the study of Mechanics we use the term in its ordinary significance.

If we do wish to give a definition of work, we can say that *work is done by a force when it causes a body to move.* The essential points to notice are (i) that we cannot have work done without the action of a **force,** and (ii) that no force does work by merely acting upon a body: it must give **motion** to the body.

The amount of work done is measured by the product of the magnitude of the force and the distance through which it acts, that is,

$$\textbf{work done} = \textbf{force} \times \textbf{distance.}$$

In using this result, however, there are two important rules that we must take care always to follow. Firstly, if the force is a *variable* one, then we must take its *mean* value in order to calculate the work done. Secondly, we must always measure the distance *in the same direction as the direction of the force.* This is a very important point to notice, because the direction of motion of the *body* is not always the same as the direction of the *force* which causes the motion, on account of the action of other (passive) forces.

Work is a *scalar quantity*: the force which does the work has, of course, direction, but work itself has magnitude only: there is no idea of direction involved.

MEASUREMENT OF WORK. Unit work is done when unit force acts through unit distance. In the F.P.S. system the unit of force is the *poundal* and the unit of distance is the *foot*: therefore unit work is done when a force of one poundal acts through a distance of one foot. This unit of work is termed accordingly the

foot-poundal. The corresponding *gravitation unit* of work is the **foot-pound,** which is equal to the work done by a force of one pound-weight acting through a distance of one foot. Clearly, one foot-pound is equal to **g** foot-poundals, that is, approximately to 32·2 foot-poundals. Notice that foot-pound is really an abbreviation for *foot-pound-weight.*

In the C.G.S. system the unit of force is the *dyne* and the unit of distance is the *centimetre*: therefore unit work is done when a force of one dyne acts through a distance of one centimetre. This unit of work is termed an **erg.** The corresponding *gravitation unit* of work is the **gramme-centimetre,** which is equal to the work done by a force of one gramme-weight acting through a distance of one centimetre. One gramme-centimetre is obviously equal to **g** ergs, that is, approximately to 981 ergs.

Both the foot-poundal and the erg are very small quantities of work. A foot-poundal is approximately equal to the amount of work done when a body weighing half-an-ounce is lifted a vertical distance of one foot, which is not very much! An erg is, however, very much less, for one foot-poundal is approximately equal to 420,000 ergs, so that an erg is extremely minute. In consequence of this excessive smallness, the **joule** is sometimes used instead of the erg. One joule is equal to ten million ergs, and is very nearly equal to three-quarters of a foot-pound.

DIMENSIONS OF WORK. We have seen that the work done by a force is equal to the product of the force and the distance through which it acts. Therefore the dimensions of work will be obtained by multiplying the dimensions of force by the dimensions of distance or length. The dimensions of force are $[M][L][T]^{-2}$, and those of length are $[L]$. The dimensions of work are therefore $[M][L][T]^{-2}$ multiplied by $[L]$, that is $[M][L]^2[T]^{-2}$.

It will be seen that the dimensions of work are

exactly the same as those we have previously obtained as the dimensions of *torque* or the *moment of a force*. The two quantities **work** and **torque** are, however, essentially different in nature. The reason why their dimensions are the same is that each quantity is the product of a *force* and a *distance* : the distinction between them lies in the *direction* in which the distance is measured.

Torque = force × distance **perpendicular** to direction of force.
Work = force × distance in **same** direction as force.

We shall see later that torque multiplied by angle of displacement gives work done and we know that an angle has no dimensions.

WORK DONE AGAINST GRAVITY. Work is done *by* a force when the motion which takes place is due to the action of that force. We speak also of the work done *against* a force : this is the work done in overcoming the resistance of a force which opposes motion. Work can only be done **by** an **active** force : but work may be done **against** either an **active or passive** force.

For example, we speak of the **work done against gravity.** Gravity, or the attraction of the earth, is, as we know, a force which always acts vertically downwards, that is, towards the centre of the earth, and it acts on all bodies. If, then, we lift a body from some position to some higher position, we do so against the resistance of the force of gravity, and therefore do work in overcoming that resistance. The work done is equal to the product of the force and the distance, i.e.,

Work done against gravity = weight × height.

If we lift a body, whose weight is W poundals, through a height of h feet, then the work done against gravity is Wh foot-poundals, that is, Mgh foot-poundals, where M is the mass of the body in pounds. It should be noticed that, as we always, in computing the amount of work done, measure the distance in the same direc-

tion as that of the force, therefore in determining the amount of work done against gravity, we must take the **vertical** distance, even though we may actually move the body to its higher position by a path which is not vertical. It is only the *vertical distance moved* that counts in measuring the work done against gravity.

For example, if a body were moved from some position A, to some other position B, as indicated in Fig. 105, then the work done against gravity would be equal to the weight Mg, of the body, multiplied by the height h which is the *vertical* distance between the two positions, quite irrespective of the route by which the change of position took place. It might be lifted vertically to the position C and then moved horizontally to B, or it might be moved horizontally to D and then lifted to B, or it might be moved diagonally from A to B. So far as the work done against gravity is concerned, the only thing that matters is the vertical height through which the body has been raised.

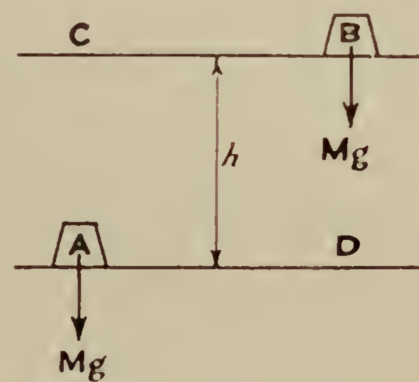


FIG. 105.

WORK DONE AGAINST FRICTION. We have another important case of work done *against* the resistance of a force: that is the work done against *friction*. As we have already seen, friction is a passive force which always opposes motion: therefore, if a moving body is acted upon by friction at all, then its motion is opposed by that force of friction, and work must be done in overcoming that resistance.

In all real cases of moving bodies with which we have to deal in engineering and general affairs, there is *some* amount of friction, although it may be very small. Therefore in *all* such cases some amount of work must be done against friction, and in many cases the amount of work expended in this way is very considerable.

Here, again, we must impress on our minds that the

work done against friction is the product of the force of friction and the distance moved *measured in the direction of that force.*

For example, if a block of wood is dragged up a plane inclined at an angle α to the horizontal, as shown in Fig. 106, by a force which does not act either vertically or along the plane, but at some angle β to the latter, then we may consider the work done in two different ways. We may first consider the work done *by* the active force, P; this will be equal to the product of P and the distance through which it acts, that is, the distance moved *measured in the direction of* P. If the distance moved along the plane by the body is s feet, then the work done by P will be equal to ($P \times s.\cos\beta$) foot-poundals, P being measured in poundals: this will be the whole amount of work done. The other way is to consider the work done *against* the forces which oppose motion, that is, in this case, the work done against gravity and against friction. The first of these is equal to the weight of the body multiplied by the vertical distance moved, that is to ($Mg \times s.\sin\alpha$) foot-poundals. The second is equal to the force of friction F, multiplied by the distance moved measured in the direction of the force, that is along the plane: it is therefore equal to ($F \times s$).

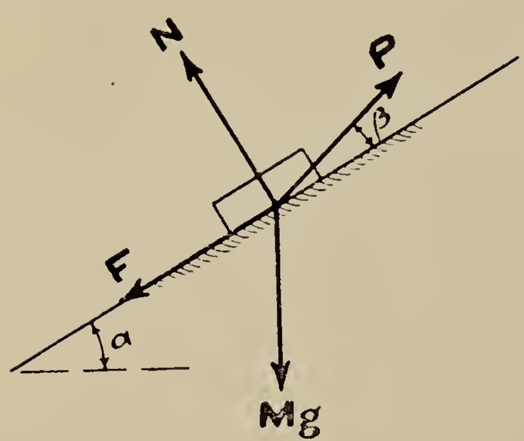


Fig. 106.

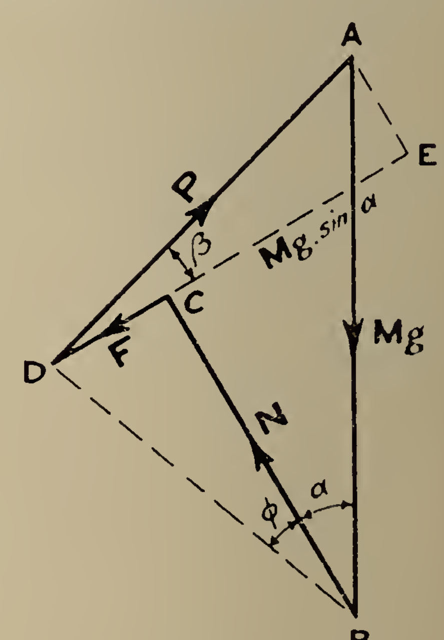


Fig. 107.

Assuming that the body moves with uniform speed, the work done *by* P must be equal to the work done *against* Mg and F. Therefore we have :—

$$P.s.\cos\beta = Mg.s.\sin\alpha + F.s.$$

whence

$$P.\cos\beta = Mg.\sin\alpha + F.$$

Reference to the force diagram shown in Fig. 107 will confirm this.

This example is worth careful study: a thorough understanding of it will provide the key to the solution of many problems, and will help to fix on the mind some very important principles.

CENTRIPETAL FORCE. When a body moves along a circular path, it does so, as we have seen, under the compulsion of a force which is always directed towards the centre of the circle, and which therefore acts perpendicularly to the direction in which the body is moving at any instant, as indicated in Fig. 108. Consequently the distance moved by the body *in the direction of the centripetal force* is always zero: therefore the work done by this force is zero. In other words, *centripetal force does no work.*

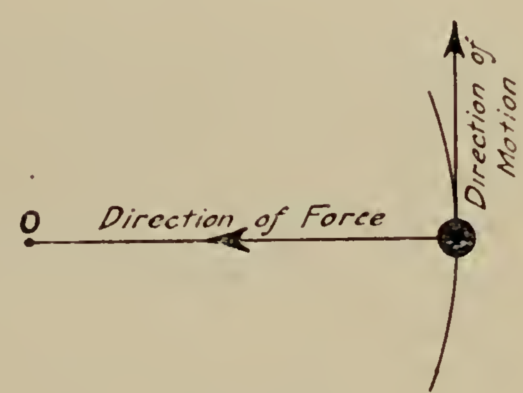


FIG. 108.

ENERGY. We use the term **energy** in Mechanics in much the same sense as in ordinary affairs, though perhaps with more exactness. Energy is defined as the *capacity for doing work.* We say that a body possesses energy when by reason of its position or condition it is capable of doing work. Energy is clearly a *scalar quantity*: it takes no account of direction.

Energy exists in the universe in many different forms, some of them so unlike that at first sight it might seem hardly credible that they are really only varying forms of the same thing and not fundamentally different quantities. The study of these forms in general belongs rather to the domain of Physics than to that of Mechanics, but we must have a clear idea of the nature and principal properties of energy generally, before we can deal satisfactorily with those forms of it which most concern us in the study of Mechanics. Let us glance at some of the aspects in which we meet with energy.

Chemical energy is stored in most substances in a greater or less degree. It is the quality which makes some substances so much more active than others, under suitable conditions. Keeping in mind the general definition of energy as the capacity for doing work, we may say that a body possesses chemical energy when by reason of the composition of its material it is capable of doing work. This form of energy is not usually evident to the senses when we examine a piece of material: we cannot tell by mere inspection whether the body contains much or little chemical energy. For example, explosives contain a very large amount of chemical energy stored in them, but this is not perceptible to the senses, and remains latent and inactive until the explosion takes place, when it is released with great suddenness and a large amount of work is done in a very short time.

The value of *fuels* lies in the amount of energy they have stored in them, ready to be released by combustion with the air or its oxygen. They give us another instance of chemical energy.

Heat is another form in which we constantly encounter energy. This differs widely from the previous case, for heat is, in general, readily perceptible by the senses. When we utilise chemical energy we most often do so by first converting it into heat. The sun also provides us with a very large amount of heat, and is indeed the principal source of supply of energy to the earth.

Electrical energy is another form. The electric current is brought to our workshops and homes, and there does much useful work for us. In this form of energy there is little obvious resemblance to either of the previous varieties.

Light is a form of energy which in some ways resembles heat and in others electricity. Here, again, we receive our principal supply from the sun, by means of radiation.

We now have to consider three forms of energy which are of more direct interest to us in the study of Mechanics.

Potential energy is the capacity which a body has for doing work owing to its *position* and to the action upon it of forces of attraction or repulsion. The most important of these, from our point of view, is the attraction exerted by the earth on all material bodies ; that is, the force of *gravity*. For example, a boulder at the top of a cliff possesses potential energy : if it falls it will do a considerable amount of work on anything—or anybody—that happens to be at the foot of the cliff below it.

Similarly, bodies which are separated and are magnetic, or electrically charged, have potential energy by reason of the attraction or repulsion between them.

Strain energy is the capacity which a body possesses for doing work by reason of its size or shape having been changed by some means, and by reason of its **elasticity** which enables it to regain its size or shape, more or less completely, doing work in the process. An example of this is to be seen in any ordinary watch or clock, where by the process of *winding-up* we put work into the mainspring which is stored in the form of strain energy, and is afterwards given out again in the process of making the watch or clock work, i.e., in causing the wheels to rotate and the hands to move round and round.

Strain energy is frequently grouped with potential energy as though there were no distinction between them. They are, however, essentially different forms. Potential energy depends upon something outside the body altogether, viz., the force of gravity : strain energy, on the other hand, depends on elasticity, which is a property of the material of which the body is composed, and has nothing whatever to do with gravity. To confuse such different varieties is inexcusable.

Kinetic energy is the capacity of a body to do work by reason of its *motion*. We have seen that a body can only be set in motion by the action of force upon it and that the change of momentum—and hence of velocity—produced is proportional to the force and the

time during which it acts. Now during the time that the force is acting, the body is moving, and therefore work is being done on the body by the force. This work is stored in the body in the shape of *kinetic energy*. A good example of kinetic energy is to be found in a rifle bullet : lying on the table it possesses no kinetic energy, but fired from the rifle it acquires a great velocity and hence a considerable amount of kinetic energy, which it expends upon whatever it may chance to encounter.

The three forms of energy last mentioned, viz., potential energy, strain energy, and kinetic energy, are usually grouped together under the name of **mechanical energy**, and it is with these forms that we are chiefly concerned in Mechanics.

PRINCIPLE OF THE CONSERVATION OF ENERGY. All the evidence of scientific research and experience goes to show that these forms of energy, although they are so dissimilar in their characteristics, are not different things but only varieties of the same thing. We have evidence of this in the fact that we can transform one kind of energy into another kind, and we do actually make transformations of this sort every day. A lump of coal contains a considerable amount of *chemical energy* through its affinity for oxygen : by burning the coal we transform this energy into *heat*. If we use this heat to turn water into steam and so to drive an engine we can obtain from it *kinetic energy*. This kinetic energy can be used to drive a dynamo and generate *electrical energy*, which in turn can be utilised to provide energy in the forms of either *light* or *heat*. These changes can be continued in great variety, passing from one kind of energy to another and back again, but in every case we find that we must have energy to start with : we cannot create it. We shall see also that we cannot destroy energy, although this is perhaps rather more difficult to realise.

These results are embodied in a law which must be regarded as one of the most important in the whole of

physical science, and is fundamental to much of our work in Mechanics.

The **Principle of the Conservation of Energy** states that *energy can neither be created nor destroyed, although it can be transformed from one kind to another.*

There are cases in which energy does seem to be destroyed, but if we investigate such cases carefully, we find that the energy which has disappeared has really only been transformed into some other variety. For example, if we turn heat into electrical energy and then back again into heat, we find that the quantity of heat which we have at the end is less than the quantity with which we started. This does not, however, prove that any of the energy has been destroyed, but simply that we have lost some of it in the various processes ; just as, if we transfer water from one vessel to another and back again, we inevitably lose a little through spilling or evaporation or wetting the sides of the vessels. We can never entirely eliminate losses of heat by radiation, conduction, and convection, losses of electricity by slight defects of insulation, and so on.

Sometimes it has seemed that people have discovered some means of creating energy, but always further investigation has shown that they have, in reality, merely discovered another source from which to draw energy that was already in existence. In fact, the more closely the matter is investigated, the more confirmation we find for this principle.

DISSIPATION OF ENERGY. All mechanical processes and all physical life depend on the possession by some bodies of more energy than is possessed by others. If all bodies, of every kind, were reduced to the same level of energy, then no motion or life of any sort would be possible. For instance, heat will only pass from a hotter body to a cooler one : no matter how much heat there may be in a body, it is of no use to us unless we can find a cooler body into which to pass it. Similarly, water will only flow under the action of gravity when by

doing so it can reach a lower level. We cannot work a water-wheel merely because we have plenty of water: we must also have the possibility of the water falling from a higher to a lower level if we are to utilise it to turn the wheel.

As we have seen, although energy cannot be destroyed, it can be lost, so that we can no longer make use of it. As a rule this lost energy disappears into space in the form of heat: this is part of a general tendency for all forms of energy in the universe to *degrade* to heat, and to come down to the same level, that is, to the same temperature. This process is known as the **dissipation of energy** or the **degradation of energy.** When it is complete we shall have in the universe no mechanical energy, no chemical energy, no electrical energy; nothing but heat, and that all at the same level of temperature. All forms of life and activity will then be extinct. This does not mean that there will be any less energy in the universe than there is to-day, but simply that it will all be at the same level and all therefore useless.

We may add that this process will require, at least, some millions of years for its completion.

POTENTIAL ENERGY. We will now consider in more detail those forms of energy which most nearly concern us in the study of Mechanics. The first of these is *potential energy.*

We have seen that the amount of potential energy possessed by a body depends upon the position of the body and upon the force with which gravity, or any similar attraction, acts upon it. Let us find an expression for the amount of potential energy possessed by a body under the action of gravity.

Suppose that a body, of mass M pounds, is at a height h feet above the ground. Then the force with which the earth attracts the body, that is, its weight, is Mg poundals. Now if the body falls to the ground, the work done on the body by the force of gravity will be equal to the product of the force and the distance through which

it acts, that is, it will be Mgh foot-poundals. This work can be given out again by the body, so that the body is capable of doing Mgh foot-poundals of work. But the capacity of a body for doing work is what we have termed its energy: therefore the energy of the body in this case is Mgh foot-poundals, and that is the general expression for the potential energy of any body.

We measure energy, then, in exactly the same units as we employ for measuring work, and, of course, this will be equally true whether we are using absolute or gravitation units, and whether we are employing the F.P.S. or the C.G.S. system. We may, in fact, regard work and energy as the same quantity under different conditions :—

Energy is *stored work.*
Work is *energy utilised.*

KINETIC ENERGY. We have defined *kinetic energy* as the capacity of a body to do work by reason of its motion. Let us find an expression for this form of energy. Suppose that we have a body of mass M pounds, moving from rest under the action of a constant force P poundals. Then the body receives an acceleration of a feet per second per second, and the magnitude of the force is $P = M.a$ poundals. In t seconds the body will have acquired a velocity of v feet per second, where $v = a.t$. The *mean* velocity during this period will clearly be $\frac{1}{2}v$ feet per second, and the distance travelled will be therefore equal to the mean velocity multiplied by the time, that is, it will be $\frac{1}{2}v \times t$ feet, or $\frac{1}{2}a.t^2$ feet.

The kinetic energy of the body, at the end of t seconds from rest, will be equal to the work done upon it during that period, that is, it will be equal to the product of the force acting and the distance moved, so that we have :—

$$\begin{aligned}\text{Kinetic energy} &= \text{work done on body}\\ &= \text{force} \times \text{distance}\\ &= M.a \times \tfrac{1}{2}a.t^2\\ &= \tfrac{1}{2}M.a^2.t^2\\ &= \tfrac{1}{2}Mv^2 \text{ foot-poundals.}\end{aligned}$$

COMPARISON OF MOMENTUM AND KINETIC ENERGY. If a body, initially at rest, is set in motion by the action of a force, then we have :—

Momentum of body

= mass × velocity imparted
= mass × acceleration × time from rest
= force × time from rest,

whence Force $= \dfrac{\text{momentum imparted}}{\text{time occupied}}$;

also **Kinetic energy of body**

= work done on body
= force × distance travelled

$= \dfrac{\text{momentum}}{\text{time}} \times$ distance travelled

$=$ momentum $\times \dfrac{\text{distance travelled}}{\text{time occupied}}$

= **momentum** × **mean velocity**
= momentum × $\frac{1}{2}$ final velocity
$= Mv \times \frac{1}{2}v$
$= \frac{1}{2}Mv^2$ as before.

The reason why the *square* of the velocity enters into the expression for kinetic energy will be understood when it is realised that an increase in the velocity of a body means an increase both of the *force* which can be exerted by the body and of the *distance* through which it can exert that force; for, as we have seen, the product of the force and the distance through which it acts gives the amount of the work done.

Putting it in another way, we may say that if a body is brought to rest in a given time t, then its momentum is destroyed in that time, and the *force* it exerts during that time, being equal to the rate of change of momentum, is Mv/t. The mean velocity of the body during the same interval is $\frac{1}{2}v$, and the *distance* through which it moves is therefore $\frac{1}{2}vt$.

Therefore the work it does in coming to rest

$$= \text{force exerted} \times \text{distance travelled}$$

$$= \frac{Mv}{t} \times \tfrac{1}{2}vt$$

$$= \tfrac{1}{2}Mv^2 \qquad \text{as before.}$$

We have considered this at some length because students often find difficulty in comprehending the relationship between *momentum* and *kinetic energy*, and we cannot afford to have confused ideas on such an important part of the subject. Let us now sum up the results we have obtained.

Suppose that a body of mass M pounds moves from rest under the action of a constant force P poundals, which gives it an acceleration of a feet per second per second. Suppose also that at the end of t seconds, it has moved through a distance s feet, and attained a velocity of v feet per second. Then we have :—

Kinetic energy imparted = **Force** × **distance**

$$= P.s$$
$$= M.a.s$$
$$= M.a.\ \tfrac{1}{2}vt$$
$$= \tfrac{1}{2}\mathbf{Mv^2} \text{ foot-poundals.}$$

Momentum imparted = **Force** × **time**

$$= P.t$$
$$= M.a.t$$
$$= \mathbf{Mv} \text{ F.P S. units.}$$

It will be seen that the *momentum* of a body gives us a measure of its *capacity for exerting force* for a given time, and that the *mean velocity* of a body gives us a measure of its *capacity for change of position* in a given time, so that the product of the *momentum* of a body and its *mean velocity* gives us its capacity for doing work, that is, its *kinetic energy*.

MACHINES. A **machine** may be defined as *a piece of apparatus for changing the direction or the magnitude of a force or both.* In Mechanics we are not con-

cerned particularly with the details of construction of machines, but with the principles on which they work.

The object of using machines is to enable us to adapt the forces at our disposal to suit the resistances which we wish to overcome. The forces may be those exerted by men or animals, or may be obtained by utilising the energy stored in fuel and other natural sources of supply. The resistances may be those of gravity, or friction, or the cohesive properties of material.

One important class of machines is that intended to facilitate the lifting of heavy bodies. In such machines, in general, a small force applied to the machine, and known as the **effort,** enables us to raise a heavy body against the resistance of the force of gravity. The weight of the body lifted is termed the **load,** and as a rule the load is much greater than the effort.

This result is achieved by arrangements differing greatly in detail, but resembling each other in the fact that the *distance* moved by the effort is much greater than the distance moved by the load. The general principle involved is that of the Conservation of Energy, but it is convenient in this connection to state it in a special form which is termed the Principle of Work.

PRINCIPLE OF WORK. *The work obtained from any machine is equal to the work put into the machine, less any spent in overcoming friction.*

If we could entirely eliminate friction from a machine, then the work got out of the machine would be exactly equal to the work put into it. In the case of real machines there is always some amount of friction to be overcome, and therefore the work got out is always less than the work put in. The difference between the work put in and the work got out may be regarded as *lost work,* because it is not usually available for any useful purpose, but the energy dissipated in this way is, of course, not destroyed but merely converted into some other form, such as heat. In order to operate machinery as economically as possible, we therefore pay great attention to the

reduction of friction, but we can never abolish it altogether.

VELOCITY RATIO AND MECHANICAL ADVANTAGE. The ratio of the *distance moved by the effort* to the *distance moved by the load*, is known as the **velocity ratio** of the machine.

The ratio of the magnitude of the *load* to the magnitude of the *effort*, is termed the **force ratio** or the **mechanical advantage** of the machine. The greater the mechanical advantage of a machine, the greater is the load that can be moved by means of a given effort.

The work put into a machine is the product of the *effort* and the *distance* through which it moves. The work got out of a machine is the product of the *load* and the distance through which it is moved. If there were no friction, then these two quantities would be exactly equal and the mechanical advantage, or force ratio, would be equal to the velocity ratio. As in real machines there is always some friction, so the mechanical advantage is always less than the velocity ratio.

The velocity ratio is a constant quantity, as a rule, for any particular machine, but the mechanical advantage varies with the load which is being dealt with. Usually the mechanical advantage increases as the load increases.

EFFICIENCY. As a general rule we want to get out of a machine as high a proportion as possible of the work that we put into it, and the better we succeed in this the more efficient we consider the machine to be. Therefore we say that *the* **efficiency** *of a machine is the ratio of the work obtained from the machine to the work put into it.* It is therefore equal to :—

$$\frac{\text{Load} \times \text{distance moved by load}}{\text{Effort} \times \text{distance moved by effort}}$$

which is clearly equal to :—

$$\frac{\text{Mechanical advantage}}{\text{Velocity ratio}}$$

Efficiency is usually denoted by the Greek letter η

(eta) and is generally expressed as a *percentage*. Evidently, it will always be less than 100 per cent.

If we experiment with any lifting machine, such as a screw-jack for example, we find that the effort required increases by approximately equal amounts for equal increments of load, so that if we plot, on squared paper, the corresponding values of effort and load, we obtain a straight line to show the relation between them, as indicated in Fig. 109.

If there were no friction in a machine, then the load which could be moved by it would be equal to the product of the *effort* and the *velocity ratio*, so that if we denote the effort by P and the velocity ratio by V, we may say that the *theoretical load* that could be moved if there were no friction would be equal to PV. We may denote the actual load, corresponding to the effort P, by W, so that the difference between the theoretical load and the actual load is PV — W. This we may term the *loss of load owing to friction*. If we plot this against the load, on graph paper, we shall again obtain a straight line to show the relation between the two quantities: see Fig. 109.

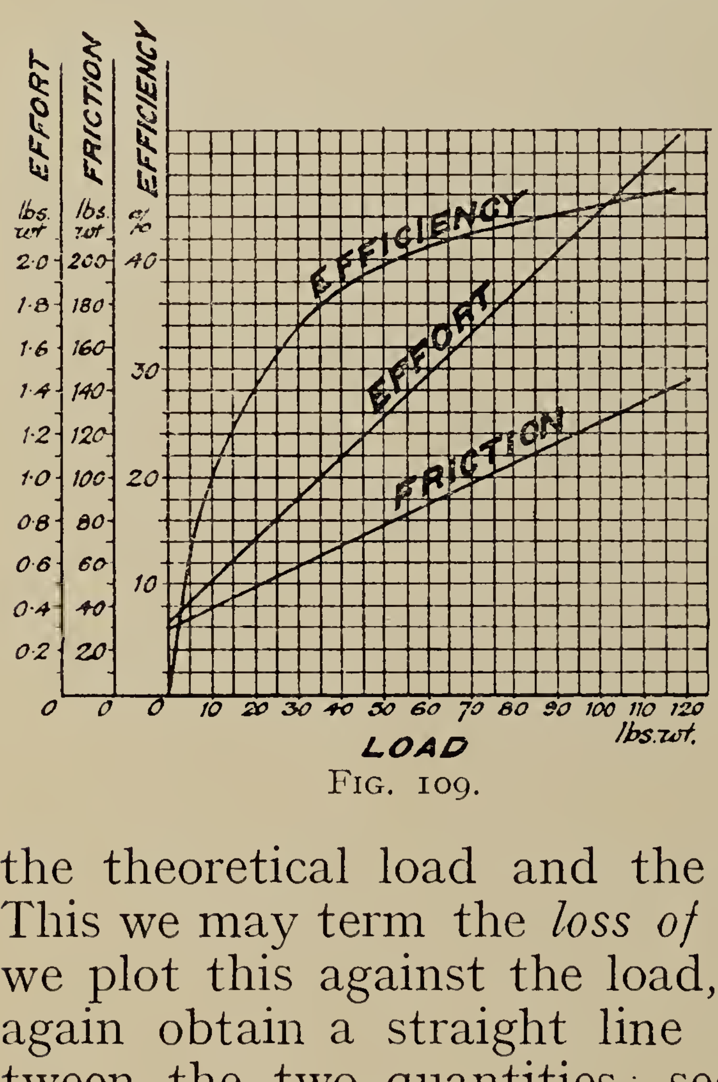


Fig. 109.

We have seen that we can express the efficiency of a machine in more than one way already. Yet another way to express it is to take it as the ratio of the actual load to the theoretical load. Then we have $\eta = W/PV$.

If we plot the efficiency of the machine against the load we do not get a straight line but a curve to express the relationship. With any machine, if we have no load at all, we shall still find that we require some effort

to move the machine, because there is always some frictional resistance to overcome. Clearly, then, the efficiency of the machine for zero load will be zero.

POWER. This is another case where we use an everyday expression in Mechanics in a strictly limited sense, and we must therefore take particular care that we do not employ it in any of the other ordinary uses, for they are not permissible in Mechanics in any circumstances.

We define **power** as the *rate of doing work.* If work is being done uniformly, then the power is the work done per unit time, and is equal to the total work done, divided by the time occupied in doing it.

The unit of power in the F.P.S. system is *one foot-poundal per second.* The unit in the C.G.S. system is *one erg per second.* The corresponding gravitation units are *one foot-pound per second,* and one *centimetre-gramme per second,* but these are not much used. The practical unit of power, which is the one of most importance, is the **horse-power.** One horse-power is equal to 33,000 *foot-pounds per minute,* or 550 foot-pounds per second. Another practical unit of power is the **watt** which is a rate of doing work equal to *one joule per second.* This unit of power is much used in connection with electrical energy, where also a still larger unit, termed a **kilowatt,** and equal to 1,000 watts, is in use. We may note that 746 watts equal one horse-power.

DIMENSIONS. Energy and work being interchangeable quantities, the dimensions of energy are the same as those of work, viz., $[M][L]^2[T]^{-2}$. Power being rate of doing work, the dimensions of power are obtained by dividing the dimensions of work by the dimensions of time: they are therefore $[M][L]^2[T]^{-2}$ divided by $[T]$, that is $[M][L]^2[T]^{-3}$.

SUMMARY OF CHAPTER 12

Work is done by a force when it causes a body to move. The amount of work done is measured by the product of the force and the distance through which it acts. Work is a scalar quantity.

Unit work is done when unit force acts through unit distance. The F.P.S. unit of work is therefore the **foot-poundal**, and the C.G.S. unit is the **erg**, which is the work done by a force of one dyne acting through a distance of one centimetre. The work done by a force of one pound-weight acting through a distance of one foot is termed a **foot-pound**, and is equal to **g** foot-poundals. One **joule** is equal to ten million ergs.

The dimensions of work are $[\mathbf{M}][\mathbf{L}]^2[\mathbf{T}]^{-2}$.

Work done against gravity = weight × height.

Centripetal force does no work.

Energy is capacity for doing work : it is a scalar quantity. Energy exists in the universe in many different forms, such as Chemical Energy, Heat, Light, Electrical Energy, and Mechanical Energy. The latter form includes Potential, Strain, and Kinetic Energy.

The **Principle of the Conservation of Energy** states that energy can neither be created nor destroyed, although it can be transformed from one kind to another.

The **potential energy** of a body is equal to its weight multiplied by its height above some given level.

The **kinetic energy** of a body is equal to half the product of its mass and the square of its velocity.

Kinetic energy = force × distance.

Momentum = force × time.

Energy is stored work : work is energy utilised.

A **machine** is a piece of apparatus for changing the direction or the magnitude of a force or both. Machines enable us to adapt the forces at our disposal to suit the resistances which we wish to overcome.

The **Principle of Work** states that the work obtained from any machine is equal to the work put into the machine, less any spent in overcoming friction.

The ratio of the distance moved by the **effort** in a machine to the distance moved by the **load** is known as the **velocity ratio** of the machine. The ratio of the load to the effort is termed the **force ratio** or the **mechanical advantage** of the machine.

The **efficiency** of a machine is the ratio of the work obtained from the machine to the work put into it.

$$\text{Efficiency} = \frac{\text{load} \times \text{distance moved by load}}{\text{effort} \times \text{distance moved by effort}}$$
$$= \frac{\text{mechanical advantage}}{\text{velocity ratio}}$$
$$= \frac{\text{actual load}}{\text{theoretical load}}.$$

The efficiency of a machine must always be less than 100 per cent.

Power is the rate of doing work. The systematic units of power are *one foot-poundal per second*, and *one erg per second*. One **horse-**

power is equal to 33,000 foot-pounds per minute. One **watt** is equal to one joule per second. One **kilowatt** is equal to one thousand watts. 746 watts equal one horse-power.

The dimensions of energy are the same as those of work. The dimensions of power are $[M][L]^2[T]^{-3}$.

EXAMPLES XII

(For Hints on Working Examples, see page 21.)

1. *Find the work done when a body of 2·7 tons mass is lifted a vertical distance of 15 inches.*

Work done = force × distance
= weight × vertical height
= (2·7 × 2,240) pounds-weight × $1\frac{1}{4}$ feet
= 7,560 foot-pounds.

2. *A train of 280 tons mass travels at a speed of 55 miles per hour. The tractive resistance at that speed is 15 pounds-weight per ton. Determine (a) the momentum of the train in foot-ton-second units, (b) the kinetic energy of the train in foot-tons, and (c) the horse-power exerted by the engine.*

55 miles per hour

$$= \frac{55 \times 22}{15} \text{ feet per second}$$

= 80·7 feet per second.

Momentum
= mass × velocity
= 280 tons × 80·7 feet per second
= 22,600 foot-ton-second units.

Kinetic energy

$$= \tfrac{1}{2} \text{ mass} \times \text{square of velocity}$$

$$= \tfrac{1}{2} \times 280 \times 2{,}240 \times 80{\cdot}7 \times 80{\cdot}7 \text{ foot-poundals}$$

$$= \frac{\tfrac{1}{2} \times 280 \times 80{\cdot}7 \times 80{\cdot}7}{32{\cdot}2} \text{ foot-tons}$$

= 28,260 foot-tons.

Horse-power

$$= \frac{\text{force in pounds-weight} \times \text{velocity in feet per second}}{550}$$

$$= \frac{15 \times 280 \times 80{\cdot}7}{550} = 616.$$

3. *A block of wood, of 35 pounds mass, is dragged up a plane, 12 feet in length, inclined at an angle of 20 degrees to the horizontal. The coefficient of friction between the block and the plane is ·28. Find the force required and the work done (a) if the direction of the force is parallel to the plane, and (b) if the direction of the force is horizontal.*

CASE (*a*).

The forces acting on the block are :—

(i) Its weight W vertically downwards.
(ii) The reaction N of the plane, normal to the plane.
(iii) The force of friction F, opposing motion and therefore down the plane.
(iv) The force P required to maintain motion, acting up the plane.

The forces F and N may be combined to give us the single force R, as shown in Fig. 110. We then have the three forces W, R, and P acting on the block and keeping it in equilibrium, and as they are not all parallel they must meet at a point. They can therefore be represented by the sides of a triangle, taken in order, as shown.

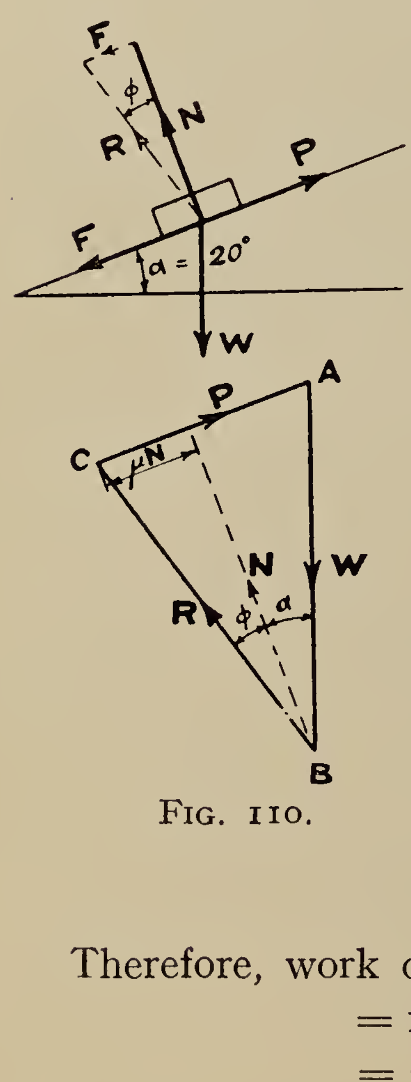


FIG. 110.

In order to obtain the magnitude of P we can either draw the triangle ABC to scale (as large as possible) and then measure the length CA, or we can find the value by calculation.

If we insert in the triangle the dotted line, as shown, to represent the normal reaction N, then it is clear that :—

$$
\begin{aligned}
P &= F + W.\sin\ \alpha \\
&= \mu.N + W.\sin\ 20^\circ \\
&= \mu.W.\cos\ 20^\circ + W.\sin\ 20^\circ \quad \text{(since clearly } N = W.\cos\ 20^\circ\text{)} \\
&= W(\mu.\cos\ 20^\circ + \sin\ 20^\circ) \\
&= 35 \text{ pounds-weight } \{(\cdot 28 \times \cdot 940) + \cdot 342\} \\
&= 35 \text{ pounds-weight } \times \cdot 605 \\
&= 21\cdot 18 \text{ pounds-weight.}
\end{aligned}
$$

Therefore, work done :—

$$
\begin{aligned}
&= \text{force} \times \text{distance} \\
&= 21\cdot 18 \text{ pounds-weight} \times 12 \text{ feet} \\
&= 254\cdot 1 \text{ foot-pounds.}
\end{aligned}
$$

We shall obtain the same result if we determine separately the work done against friction, and the work done against gravity, and add them together.

Force of friction

$$= \mu.N$$
$$= \mu.W.\cos 20°$$
$$= \cdot 28 \times 35 \text{ pounds-weight} \times \cdot 94$$
$$= 9\cdot 22 \text{ pounds-weight.}$$

Work done against friction

$$= \text{force} \times \text{distance}$$
$$= 9\cdot 22 \text{ pounds-weight} \times 12 \text{ feet}$$
$$= 110\cdot 6 \text{ foot-pounds.}$$

Work done against gravity

$$= \text{weight} \times \text{vertical height}$$
$$= 35 \text{ pounds-weight} \times (12 \text{ feet} \times \sin 20°)$$
$$= 35 \text{ pounds-weight} \times (12 \times \cdot 342) \text{ feet}$$
$$= 143\cdot 7 \text{ foot-pounds.}$$

Total work done

$$= 110\cdot 6 \text{ foot-pounds} + 143\cdot 7 \text{ foot-pounds}$$
$$= 254\cdot 1 \text{ foot-pounds,}$$

as before.

CASE (*b*).

We have again four forces, W, N, F, and P, acting upon the body, but P is now horizontal, and therefore the magnitudes of N, F, and P will all be changed.

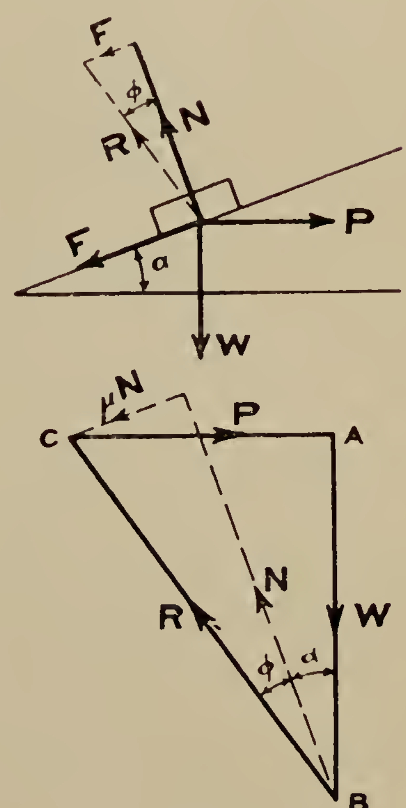


FIG. 111.

Let $\phi = \text{angle of friction}$

$$= \tan^{-1} F/N$$
$$= \tan^{-1} \mu$$
$$= \tan^{-1} \cdot 28.$$

Then evidently

$$P = W.\tan(\phi + a)$$
$$= W \frac{\tan \phi + \tan a}{1 - \tan \phi.\tan a}$$
$$= 35 \text{ pounds-weight} \times \frac{\cdot 28 + \cdot 364}{1 - (\cdot 28 \times \cdot 364)}$$
$$= 35 \text{ pounds-weight} \times \frac{\cdot 644}{1 - \cdot 102}$$
$$= 25\cdot 1 \text{ pounds-weight.}$$

Therefore, work done

$$\begin{aligned}
&= \text{force} \times \text{distance} \\
&= 25{\cdot}1 \text{ pounds-weight} \times (12 \text{ feet} \times \cos 20^\circ) \\
&= 25{\cdot}1 \text{ pounds-weight} \times (12 \times {\cdot}94) \text{ feet} \\
&= 25{\cdot}1 \text{ pounds-weight} \times 11{\cdot}28 \text{ feet} \\
&= 283 \text{ foot-pounds.}
\end{aligned}$$

4. *In a certain lifting-machine it is found that to raise a load of* 17 *cwt. an effort of* $96\frac{1}{2}$ *pounds-weight is required, and that for each* 10 *feet moved by the effort the load rises* $3\frac{1}{4}$ *inches. Determine* (*a*) *the velocity ratio,* (*b*) *the effort required to overcome friction,* (*c*) *the loss of load due to friction, and* (*d*) *the efficiency of the machine.*

Velocity ratio

$$\begin{aligned}
&= \frac{\text{distance moved by effort}}{\text{distance moved by load}} \\
&= \frac{120 \text{ inches}}{3\frac{1}{4} \text{ inches}} \\
&= 36{\cdot}9.
\end{aligned}$$

Effort required to overcome friction

$$\begin{aligned}
&= \text{actual effort} - \text{theoretical effort} \\
&= \text{actual effort} - \frac{\text{load}}{\text{velocity ratio}} \\
&= 96{\cdot}5 \text{ pounds-weight} - \frac{17 \times 112}{36{\cdot}9} \text{ pounds-weight} \\
&= (96{\cdot}5 - 51{\cdot}6) \text{ pounds-weight} \\
&= 44{\cdot}9 \text{ pounds-weight.}
\end{aligned}$$

Loss of load due to friction

$$\begin{aligned}
&= \text{theoretical load} - \text{actual load} \\
&= (\text{effort} \times \text{velocity ratio}) - \text{actual load} \\
&= (96{\cdot}5 \text{ pounds-weight} \times 36{\cdot}9) - (17 \times 112) \text{ pounds-weight} \\
&= 3{,}560 \text{ pounds-weight} - 1{,}904 \text{ pounds-weight} \\
&= 1{,}656 \text{ pounds-weight.}
\end{aligned}$$

Efficiency of machine

$$\begin{aligned}
&= \frac{\text{actual load}}{\text{theoretical load}} \\
&= \frac{1{,}904 \text{ pounds-weight}}{3{,}560 \text{ pounds-weight}} \\
&= 53{\cdot}5 \text{ per cent.}
\end{aligned}$$

The same result may be obtained in a slightly different way :—

Efficiency of machine

$$= \frac{\text{theoretical effort}}{\text{actual effort}}$$

$$= \frac{51{\cdot}6 \text{ pounds-weight}}{96{\cdot}5 \text{ pounds-weight}}$$

$$= 53{\cdot}5 \text{ per cent.} \qquad \text{as before.}$$

5. *Calculate the horse-power required to overcome the force of gravity, when a car of* 2 *tons mass climbs a hill of* 1 *in* 16 *gradient at a uniform speed of* 9 *miles per hour.*

Nine miles per hour

$$= 9 \times \frac{5{,}280}{60} \text{ feet per minute}$$

$$= 792 \text{ feet per minute.}$$

Vertical component of this velocity

$$= 1/16 \times 792 \text{ feet per minute}$$

$$= 49{\cdot}5 \text{ feet per minute.}$$

Work done against gravity per minute

$$= \text{force} \times \text{distance per minute}$$

$$= (2 \times 2{,}240) \text{ pounds-weight} \times 49{\cdot}5 \text{ feet per minute}$$

$$= (2{,}240 \times 99) \text{ foot-pounds per minute.}$$

Horse-power required

$$= \frac{2{,}240 \times 99}{33{,}000}$$

$$= 6{\cdot}72.$$

6. Calculate the work done when a force of 45·3 poundals acts through a distance of 3 feet $7\frac{3}{4}$ inches.

7. A man, weighing 12 stone 6 pounds, goes upstairs from the ground floor to the second floor, the difference of level between the two floors being 22 feet 9 inches. Find the amount of work he does against the resistance of gravity.

8. Find the work done against gravity when a body weighing 34 cwt. is raised a vertical distance of 45 feet.

9. A locomotive exerts a draw-bar pull of 3·67 tons-weight. Find the work it does in hauling a train a distance of 23 miles. If the speed is 43 miles per hour, what is the horse-power developed by the engine ?

10. A force of 65 kilogrammes-weight acts through a distance of 37 metres. Determine the amount of work done, in joules.

11. Find the work done when a slab of iron is dragged 8 feet up a plane inclined at an angle of 25 degrees to the horizontal by a force acting parallel to the plane, if the mass of the slab is 42 pounds and the coefficient of friction between the slab and the plane is ·23.

12. A block of wood of $5\frac{1}{4}$ pounds mass is pulled up an inclined plane by a force which acts at an angle of 25 degrees to the horizontal, and at an angle of 10 degrees to the plane. If the coefficient of friction between the block and the plane is ·33, find the work done in raising the block a vertical distance of 4 feet, (*a*) when the force is more steeply inclined than the plane, and (*b*) when the plane is more steeply inclined than the force.

13. Determine the kinetic energy, in foot-tons, of a motor-car of 2,000 pounds mass, travelling at a speed of 18 miles per hour.

14. What is the kinetic energy, in F.P.S. units, of a body of 18 pounds mass moving with a velocity of 150 feet per second ?

15. A body of 84 pounds mass falls from the top of a tower 365 feet in height. Determine separately its kinetic and potential energies (*a*) at the commencement of its fall, (*b*) when it has fallen a distance of 140 feet, and (*c*) when it is five feet from the ground.

16. A train of 400 tons mass moves at a speed of 51 miles per hour against a tractive resistance of $14\frac{3}{4}$ pounds-weight per ton. Find the horse-power exerted by the locomotive. If the weight of the latter is 54 tons, of which two-thirds is on the driving wheels, what is the least coefficient of friction between the wheels and the rails ?

17. A car of $1\frac{1}{2}$ tons mass climbs a hill of 1 in 35 gradient at a speed of 25 miles per hour, the tractive resistance on the level being 25 pounds-weight per ton. What horse-power does the engine develop (*a*) in overcoming the force of gravity, and (*b*) against friction and similar resistances ?

18. In the previous question, find the kinetic energy of the car at the foot of the hill. What is the gain of energy when the car has travelled a quarter of a mile up the hill ?

19. A load of 7 pounds-weight is hung from the lower end of a helical spring and stretches it a distance of $4\frac{1}{4}$ inches. How much strain energy is stored in the spring ?

20. A bullet, of mass ·03 of a pound, moving with a speed of 500 feet per second, strikes a baulk of timber, and penetrates $1\frac{3}{4}$ inches. Find the mean force of resistance of the timber and the time occupied.

21. The momentum of a body is 256 F.P.S. units and its speed is 83 miles per hour. Determine, from first principles, its kinetic energy. If its volume is 7·8 cubic inches, what is its density ?

22. A bullet, weighing half-an-ounce and travelling at a speed

of 1,100 feet per second, penetrates 10 inches into a block of wood. Find the resisting force of the timber. If the thickness of the wood had been only 6 inches, with what speed would the bullet have emerged ?

23. A train weighing 332 tons is travelling at a speed of 45 miles per hour when the brakes are applied. What is the retarding force exerted by the brakes if the train is brought to rest in a distance of $\frac{3}{4}$ mile ? Neglect tractive resistance.

24. If the train in the previous question had been brought to rest by shutting off steam without applying the brakes, how far would it have travelled in the process ? Take the tractive resistance at a mean value of 12 pounds-weight per ton.

25. It is found in a certain lifting machine that to raise a load of $13\frac{1}{2}$ cwt. an effort of 78 pounds-weight is required, and that for each inch the load rises, the effort has to move 2 feet $7\frac{1}{2}$ inches. Determine (*a*) the velocity ratio, (*b*) the force ratio, and (*c*) the efficiency of the machine at this load.

26. In a certain lifting machine it is found that a load of half-a-ton can be raised by means of an effort of 66·5 pounds-weight. The efficiency is then 48·3 per cent. Determine (*a*) the mechanical advantage, (*b*) the velocity ratio, and (*c*) the loss of load due to friction.

CHAPTER 13: MOMENTUM AND ENERGY OF ROTATION

IN studying the motion of bodies, there are, as we have already seen, two kinds of motion to be considered, namely, **translation,** or linear motion, and **rotation,** or angular motion. For every quantity which occurs in one of these kinds of motion we shall have a corresponding quantity in the other kind. We may therefore divide the subject of motion into two parts, one concerned with translation, and the other with rotation. In Chapters 6, 8, and 12 we discussed translation: in Chapters 7 and 11 we discussed rotation.

There is another way in which we may divide the subject of motion. That part of the subject which is concerned with the properties of motion itself, without considering the forces involved, we may term **kinematics.** The other part, which deals with the forces implicated, we may term **kinetics.**

We are now in a position to make a little table of the quantities with which we must deal in the study of motion :—

	TRANSLATION.	ROTATION.
KINEMATICS	Linear Velocity Linear Acceleration	Angular Velocity Angular Acceleration
KINETICS	Linear Momentum Rate of Change of Linear Momentum Kinetic Energy of Translation	Angular Momentum Rate of Change of Angular Momentum Kinetic Energy of Rotation

It will be seen that we have already discussed the elementary principles of—

Kinematics of Translation (Chapter 6),
Kinematics of Rotation (Chapter 7),
and Kinetics of Translation (Chapters 8 and 12).

The Kinetics of Rotation we have not yet considered, with the exception of Centripetal Force. It is this part of the subject, therefore, that we now have to discuss. Before doing so, let us remind ourselves of some of the results that we have already obtained: we shall require these in our later work, so we must take care to make ourselves perfectly familiar with them. We have—

$$\text{Angular velocity} = \frac{\text{angle turned through}}{\text{time occupied}} = \frac{\text{linear velocity}}{\text{radius of rotation}}.$$

$$\text{Angular acceleration} = \frac{\text{change of angular velocity}}{\text{time occupied}} = \frac{\text{linear acceleration}}{\text{radius of rotation}}.$$

ANGULAR MOMENTUM. The quantity which we now have to consider is known by two different terms, viz., **angular momentum** and **moment of momentum**: it should be noted that these are alternative names, both referring to identically the same quantity. The former is the more generally useful term, although the latter might perhaps be considered the more logical in view of the following method of deriving it.

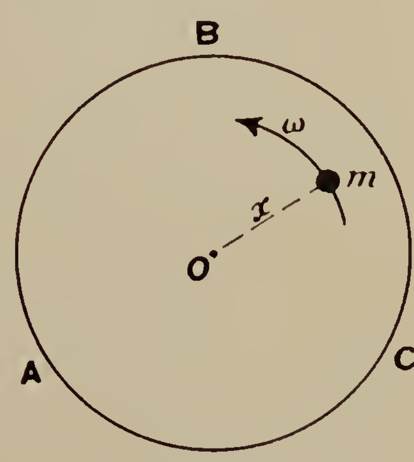


FIG. 112.

Consider a rigid body, which is rotating about a fixed axis with angular velocity equal to ω radians per second. In Fig. 112 ABC represents the body, and the axis of rotation is supposed to be perpendicular to the

plane of the paper and to pass through O. If m is the mass in pounds of any particle of the body, and x is its radius of rotation, that is, its perpendicular distance in feet from the axis of rotation, then the linear velocity of the particle at any instant is equal to ωx feet per second, and therefore the **momentum** of the particle

$$= \text{mass of particle} \times \text{its velocity}$$
$$= m.\omega.x \quad \text{F.P.S. units.}$$

The **moment of momentum** of the particle is equal to the product :—

Momentum of particle × its perpendicular distance from the axis of rotation

$$= m.\omega.x \times x$$
$$= m.\omega.x^2.$$

The moment of momentum of the whole body is equal to the sum of the moments of momentum of all the particles composing the body. Therefore, moment of momentum of whole body

$$= \Sigma m.\omega.x^2$$
$$= \Sigma mx^2.\omega$$
$$= \text{I}.\omega, \qquad \text{where I} = \Sigma mx^2.$$

MOMENT OF INERTIA. The student should pay particular attention to the quantity which we have denoted by I, that is, Σmx^2. We obtain it by finding the **second moment of mass** of each particle of the body, about the given axis, separately, and then adding them all together. This gives us the **second moment of mass** of the whole body about the given axis.

Reference back to Chapter 3 (pages 45 and 46) will remind us that the *second moment of mass* of a particle, about a given axis, is the product of the *mass* of the particle and the *square* of its perpendicular distance from the given axis, that is, $\mathbf{mx^2}$. The second moment of mass of the whole body, about the given axis, is, then, the sum of the second moments of mass of all the particles composing the body, about the given axis, that is, $\boldsymbol{\Sigma}\mathbf{mx^2}$. This quantity is, obviously, quite independent

of the speed of rotation of the body, and is, indeed, a constant for any given body with regard to any given axis. As it occurs very frequently in the kinetics of rotation, it is convenient to give it the distinctive name **second moment of mass,** and to denote it by the particular symbol **I.**

This is the best name for the quantity, for it is self-explanatory if we bear in mind the general definition of a second moment. It is not, however, the most usual name, and we must therefore learn to recognise the same quantity under the name of **moment of inertia.**

If we want a formal definition we can say that *the* **moment of inertia** (*or* **second moment of mass**) *of a body about any given axis, is the sum of the products of the mass of each particle of the body into the square of its distance from that axis*: that is :—

$$\mathbf{I} = \Sigma \mathbf{m}\mathbf{x}^2.$$

An alternative definition would follow from the general definition of a second moment, so that we could say that *the* **second moment of mass** (*or* **moment of inertia**) *of a body about any given axis, is equal to the product of the mass of the body and the square of its effective distance from that axis*: that is—

$$\mathbf{I} = \mathbf{M}\mathbf{k}^2,$$

where M is the mass of the whole body and k is its *effective* distance from the given axis. For any given body, considered with regard to a particular axis, this distance k is a constant quantity and is known as the **radius of gyration** of the body about the given axis.

MEASUREMENT OF MOMENT OF INERTIA AND RADIUS OF GYRATION. Unit moment of inertia is equal to the moment of inertia of unit mass concentrated at unit distance from the axis. In the F.P.S. system the unit of moment of inertia is therefore **one pound-foot-square,** that is, the moment of inertia of a mass of one pound at an effective distance of one foot from the axis. The moment of inertia of a body

in pound-feet-square is equal to the mass of the body in pounds multiplied by the square of the radius of gyration in feet. The C.G.S. unit is **one gramme-centimetre-square.**

The radius of gyration of a body about a given axis, being a distance, is measured in *feet* or *centimetres*, like any other length.

The determination of the values of the moment of inertia and radius of gyration, for any particular body, involves as a rule the use of the integral calculus. We shall devote the next chapter to the actual determination of these values in a number of important cases, and shall then consider the general methods of calculation which are most serviceable for that purpose. For the present, then, we will content ourselves with trying to grasp the *idea* of these two new quantities, without troubling about questions of practical calculation. It is of the utmost importance that this idea should be thoroughly mastered, for otherwise we shall find ourselves continually in difficulties when dealing with the kinetics of rotation.

DIMENSIONS OF MOMENT OF INERTIA AND RADIUS OF GYRATION. The radius of gyration of a body, about any axis, is a length, and is therefore of one dimension of length, i.e., **[L]**. The moment of inertia of a body, about any axis, being equal to the product of the mass of the body and the square of its radius of gyration about that axis, is of dimensions $[M][L]^2$.

ANGULAR MOMENTUM. We can now resume the consideration of this quantity, and may define it by stating that *the* **angular momentum** (*or* **moment of momentum**) *of a rotating body is equal to the product of the moment of inertia of the body about the axis of rotation, and the angular velocity of the body*: that is, it is equal to $I.\omega$.

We can also obtain the expression for the angular momentum of a body in another way.

We may consider that the whole mass, M, of the body is concentrated at a distance k from the axis of rotation, where k is the radius of gyration of the body about that axis.

Then the *linear momentum* of the whole body, concentrated in this way, will be, at any instant :—

$$= \text{mass} \times \text{linear velocity}$$
$$= \text{M}.\omega.k.$$

Therefore the **moment of momentum** (or *angular momentum*) of the body will be :—

$$= \text{momentum} \times \text{distance from axis of rotation}$$
$$= \text{M}.\omega.k \times k$$
$$= \text{M}k.^2\omega$$
$$= \text{I}.\omega$$

as before, since $\text{M}k^2 = \text{I}$, the moment of inertia of the body about the axis of rotation.

DIMENSIONS OF ANGULAR MOMENTUM. As we have just seen, the angular momentum of a rotating body is equal to its moment of inertia, about the axis of rotation, multiplied by its angular velocity. The dimensions of angular momentum will therefore be obtained by multiplying the dimensions of a moment of inertia by the dimensions of an angular velocity, that is, they will be $[\mathbf{M}][\mathbf{L}]^2$ multiplied by $[\mathbf{T}]^{-1}$ which gives us $[\mathbf{M}][\mathbf{L}]^2[\mathbf{T}]^{-1}$.

RATE OF CHANGE OF ANGULAR MOMENTUM. If the speed of rotation is changing, then the angular momentum of the body must be changing also, and we shall have :—

Rate of change of angular momentum (uniform or mean)

$$= \frac{\text{change of angular momentum}}{\text{time occupied}}$$

$$= \frac{\text{moment of inertia} \times \text{change of angular velocity}}{\text{time occupied}}$$

$$= \text{moment of inertia} \times \text{angular acceleration}$$
$$= \text{I}.\alpha,$$

where α (alpha) is the angular acceleration of the body in radians per second per second. *The* **rate of change of angular momentum** *of a rotating body is therefore equal to the moment of inertia of the body, about its axis of rotation, multiplied by its angular acceleration.*

We can also obtain this result in another way. Consider a particle, of mass m, at a distance x from the axis of rotation. If the speed of rotation of the body is changing, then the linear speed of the particle, and consequently its linear momentum, is changing also, and by the Second Law of (Linear) Momentum :—

Force acting upon particle

= rate of change of linear momentum,

i.e., $p = m.a$, where p is the force acting upon, and a is the linear acceleration of, the particle.

$= m.x.\alpha$, where α is the angular acceleration of the body.

Taking the **moment** of each side of the above equation about the axis of rotation, we have :—

$$p.x = m.x.^2\alpha.$$

Therefore for the whole body :—

$$\begin{aligned} \Sigma(p.x) &= \Sigma(m.x^2.\alpha) \\ &= \Sigma(m.x^2)\alpha \\ &= \mathrm{I}.\alpha \text{ as before.} \end{aligned}$$

But $\Sigma(p.x)$ is the sum of the moments of the forces acting upon the body, about the axis of rotation, that is, it is the resultant **turning moment** or **torque** on the body. We obtain, therefore, the very important result that *the rate of change of angular momentum of a body is equal to the turning moment on the body.*

LAWS OF ANGULAR MOMENTUM. In Chapter 8 we considered the Laws of Momentum as they apply to *translation* or linear motion. We now have to deal with the corresponding laws for *rotation* or angular motion. The latter, like the former, are based on the results of experiment and observation, and are therefore incapable of strict formal proof. We may verify them

by direct experiment or we may derive them from the laws of linear momentum from which they follow strictly.

First Law. *There is no change in the angular momentum of a body unless the body is acted upon by some external torque.*

Second Law. *When there is a change of angular momentum, then the torque producing it is proportional to the rate of change of angular momentum and acts in the same direction as the change of angular momentum.*

These laws, like those of linear momentum, are of great importance and should be studied very carefully. Let us consider them separately.

FIRST LAW OF ANGULAR MOMENTUM. This tells us that a body cannot of itself change its angular momentum. It also tells us that a single force acting through the centre of mass of a body cannot produce a change of angular momentum: we must have a *couple*. But it is necessary to remember that, as proved in Chapter 3 (page 54), a single force not passing through the centre of mass is equivalent to the same force acting at the centre of mass, *plus a couple*. Such a force will therefore produce changes in both the *linear* and *angular* momenta of the body on which it acts.

This law also gives us a means of defining **torque** in a different way from that employed in Chapter 3, for we can now say that *torque is that which produces, or tends to produce, a change of angular momentum in the body upon which it acts.*

It may be noted that the First Law of Angular Momentum is sometimes known as the **Principle of the Conservation of Angular Momentum.**

SECOND LAW OF ANGULAR MOMENTUM. This tells us that when the angular momentum of a body changes, the torque acting is proportional to the *rate* of change and is in the same direction: a result which we have already derived from the Second Law

of Linear Momentum. Summarising these two corresponding laws we have :—

Force = mass × linear acceleration
Torque = moment of inertia × angular acceleration.

These are not, of course, complete statements of the laws but are concise and convenient forms in which to remember them. *It is essential that all the laws of momentum should be both understood and remembered.*

It may be noticed here that both *angular momentum* and *rate of change of angular momentum* (or *torque*) are **vector quantities of the second class** (see Chapter 7, pages 121–123), and therefore obey the Parallelogram Law.

DIMENSIONS OF TORQUE. We have already obtained these in Chapter 8, page 147, by considering torque as the product of a force and a perpendicular distance. We can now determine them by considering torque as rate of change of angular momentum, and we shall see that we obtain exactly the same dimensions by both methods.

We may say that torque is equal to change of angular momentum divided by the time occupied : then its dimensions will be the dimensions of angular momentum divided by the dimensions of time, that is $[M][L]^2[T]^{-1}$ divided by $[T]$, which gives us the dimensions of torque as $[M][L]^2[T]^{-2}$, as before.

Alternatively we may say that torque is equal to moment of inertia multiplied by angular acceleration : then its dimensions will be the dimensions of moment of inertia multiplied by the dimensions of angular acceleration, that is $[M][L]^2$ multiplied by $[T]^{-2}$, which gives us the dimensions of torque as $[M][L]^2[T]^{-2}$, as before.

In the case of torque we have obtained the following results :—

EXPRESSION.	DIMENSIONS.
Force × perpendicular distance	$[M][L][T]^{-2} \times [L] = [M][L]^2[T]^{-2}$
$\dfrac{\text{Change of angular momentum}}{\text{time occupied}}$	$[M][L]^2[T]^{-1} \times [T]^{-1} = [M][L]^2[T]^{-2}$
Moment of inertia × angular acceleration	$[M][L]^2 \times [T]^{-2} = [M][L]^2[T]^{-2}$

KINETIC ENERGY OF ROTATION. Let ABC (Fig. 113) represent any rigid body which is rotating about an axis through O, perpendicular to the plane of the paper. Let m be the mass in pounds of any particle of the body at a distance x feet from the axis, v be the linear velocity of the particle, at any instant, in feet per second, and ω the angular velocity of the body in radians per second, so that $v = \omega x$.

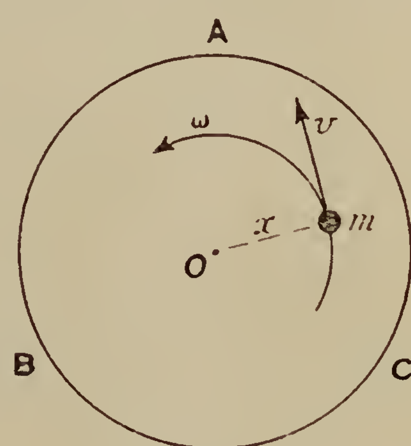


FIG. 113.

Then the kinetic energy of the particle, in foot-poundals

$$= \tfrac{1}{2}mv^2$$
$$= \tfrac{1}{2}m(\omega x)^2$$
$$= \tfrac{1}{2}mx^2.\omega^2.$$

The kinetic energy of the whole body is the sum of the kinetic energies of all the particles of which the body is composed, and is therefore:—

$$= \Sigma(\tfrac{1}{2}mx^2.\omega^2)$$
$$= \tfrac{1}{2}.\Sigma(mx^2).\omega^2$$
$$= \tfrac{1}{2}\mathrm{I}.\omega^2 \text{ foot-poundals,}$$

where $\mathrm{I} = \Sigma(mx^2)$ = the moment of inertia of the body about the axis of rotation.

We see therefore that the **kinetic energy of rotation** of a body is equal to half the product of the moment of inertia of the body about the axis of rotation and the square of the angular velocity of the body. If the moment of inertia is measured in F.P.S. units, then the kinetic

energy will be in foot-poundals, as we have seen: if the moment of inertia is measured in C.G.S. units, then the kinetic energy will be in ergs: provided that, in each case, the angular velocity is in radians per second.

DIMENSIONS OF KINETIC ENERGY OF ROTATION. Work and all forms of energy must have the same dimensions, since they are quantities essentially similar in nature. It is, however, instructive to notice how the expressions for these different forms, bearing no apparent resemblance to each other, on analysis give us the same dimensions. The student should verify this by obtaining the dimensions in each of the following cases from the expressions given:—

Work = force × distance = torque × angle (see below).

Potential energy = weight × height.

Kinetic energy of translation = $\frac{1}{2}$ mass × square of linear velocity.

Kinetic energy of rotation = $\frac{1}{2}$ moment of inertia × square of angular velocity.

WORK DONE ON A ROTATING BODY. Suppose ABC (Fig. 114) represents a body which is free to rotate about a fixed axis through O, perpendicular to the plane of the paper. Let a force P act upon the body at a distance r from the axis, and in a direction which is always perpendicular to the radius r. Then it will exert a constant torque upon the body equal to P.r.

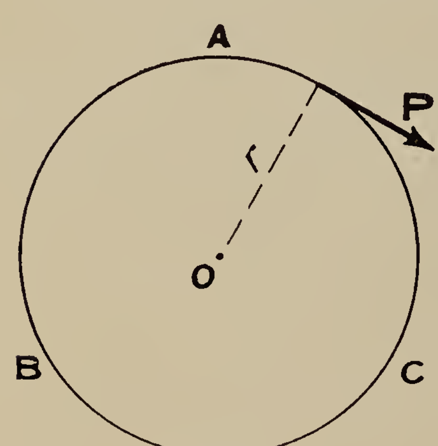


Fig. 114.

The work done in one revolution by the force P will be equal to the product of the force and the distance through which it has moved, that is, it will be equal to P × $2\pi r$, since $2\pi r$ is the length of the circular path along which the point of application of the force has moved. Now P × $2\pi r$ = P.r × 2π, that is, turning moment or torque × angle

turned through in radians. Therefore, for any number of revolutions :—

Work done = torque × angle turned through.

COMPARISON OF ANGULAR MOMENTUM AND KINETIC ENERGY. Suppose that a body, of moment of inertia I pound-feet-square, turns about an axis, from rest, under the action of a constant torque T poundal-feet, which gives it an angular acceleration of α radians per second per second. Suppose also that at the end of t seconds, it has moved through an angle θ radians, and has acquired an angular velocity of ω radians per second. Then we have :—

Kinetic energy imparted = **Torque** × **angle**

$$= \mathrm{T}.\theta$$
$$= (\mathrm{I}.\alpha.)(\tfrac{1}{2}\omega t)$$
$$= \tfrac{1}{2}\mathbf{I}\omega^2 \text{ foot-poundals.}$$

Angular momentum imparted

= **Torque** × **time**

$$= \mathrm{T}.t$$
$$= \mathrm{I}.\alpha.t$$
$$= \mathbf{I}\omega \quad \text{F.P.S. units.}$$

Let us now compare the results that we have obtained in connection with the two kinds of motion, translation and rotation. It will help to fix them in our minds if we note particularly the points in which the quantities resemble each other, and the points in which they differ from each other :—

Quantity.	Motion.	
	Translation.	Rotation.
Velocity	v	ω
Acceleration	a	α
Momentum	M.v	I.ω
Rate of Change of Momentum	M.a (force)	I.α (torque)
Kinetic Energy	$\frac{1}{2}$.M.v^2	$\frac{1}{2}$.I.ω^2

We will now consider a few special cases of kinetic energy of rotation.

DISC OR CYLINDER ROTATING ABOUT ITS OWN AXIS. This is practically what we have in the case of a solid fly-wheel. The moment of inertia of a solid disc or cylinder about its own axis is $\frac{1}{2}Mr^2$, where M is the mass of the disc or cylinder, and r is its radius. This statement we shall prove in the next chapter. If the mass and the radius are both measured in systematic units, then the moment of inertia will, of course, also be in systematic units, whether we are employing the F.P.S. or the C.G.S. system. Then we have :—

Kinetic energy of the rotating disc or cylinder

$= \frac{1}{2}.\mathrm{I}.\omega^2$

$= \frac{1}{2}.(\frac{1}{2}.\mathrm{M}.r^2.)\omega^2$

$= \frac{1}{4}.\mathrm{M}.v^2$ (foot-poundals or ergs according to system employed),

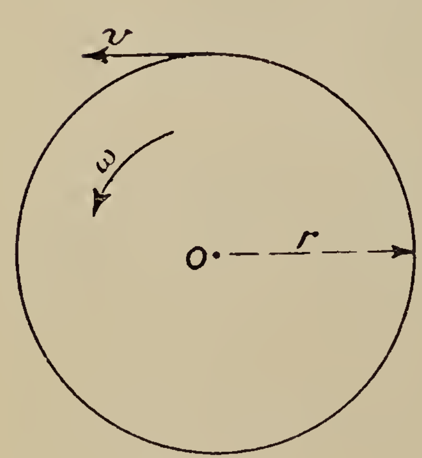


Fig. 115.

where v is the *peripheral* speed of the disc or cylinder, that is, the speed of any point on its circumference, in feet per second or centimetres per second, as the case may be.

It is interesting to note that if the *whole* disc were moving with speed v feet or centimetres per second, that is, if it were moving with linear instead of angular motion, then its kinetic energy would be twice as great, viz., $\frac{1}{2}.\mathrm{M}.v^2$. The very obvious reason for this difference is that, in the case of rotation, only points on the rim of the wheel are moving with linear speed v : all other points are moving with less linear speeds, proportionate to their distance from the axis of rotation.

DISC ROLLING DOWN INCLINED PLANE. This is a rather interesting case, because the disc is not only rotating about its own axis but is also moving along the

plane; that is, it has both motion of translation and motion of rotation.

We will assume that the disc starts from rest, and that the friction between the disc and the plane is sufficient to prevent any slipping, so that we have rolling only to consider.

Let M = mass of disc,
r = its radius,
v = its linear velocity at bottom of slope,
ω = its angular velocity at bottom of slope,
I = its moment of inertia about its own axis.

The kinetic energy of the disc at the bottom of the slope due to its motion *along the plane*, that is, its kinetic energy of translation, is:—

$$= \tfrac{1}{2}.M.v^2.$$

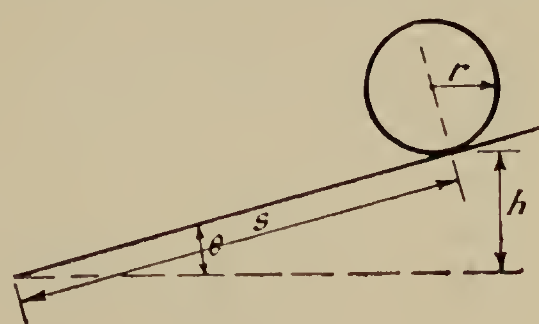


FIG. 116.

The kinetic energy of the disc at the bottom of the slope, due to its rotation *about its own axis*, that is, its kinetic energy of rotation, is:—

$$= \tfrac{1}{2}.I.\omega^2.$$

We can determine the value of ω in this case in the following way:—The disc is moving with both kinds of motion, viz., translation and rotation. These may be considered as two entirely independent motions, to the extent that if we change *one* of them by adding to it another velocity of the same kind, then *the other* will not be affected.

At the bottom of the slope, the disc *as a whole* is moving with linear velocity v, along the slope. Since there is no slipping of the disc on the plane, the point of the disc which is in actual contact with the plane must be, for the instant, at rest, so that its velocity along the plane is zero. The velocity of the centre of the disc at the same instant is v, so that the velocity of the point of the disc which is farthest from the plane, must be,

for the instant, twice as great as the velocity of the centre, that is, it must be equal to $2v$.

Now if we add to the linear velocity of *every* part of the disc a linear velocity v up the plane, then the linear velocity of each part will be changed but *the angular velocity will not be altered in any way.* The linear velocity of points at the top of the disc at any instant will now be equal to $(2v - v)$ down the slope, that is, it will be v, *down* the slope. The linear velocity of the centre of the disc will be equal to $(v - v)$, that is zero. The linear velocity of the point of the disc in contact with the plane will be $(0 + v)$, that is, v, *up* the slope. Clearly, then, the disc will be moving as though it were simply rotating about its own axis with peripheral speed v, and its angular velocity will be $\omega = v/r$ radians per second. But in giving the additional linear velocity to every part of the disc, we did not in any way change its angular velocity: therefore the actual angular velocity at the foot of the slope must be $\omega = v/r$ radians per second.

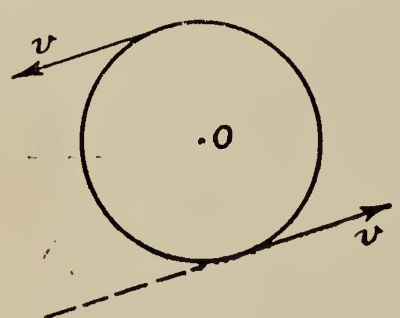


Fig. 117.

Therefore the kinetic energy of rotation of the disc at the bottom of the slope

$$= \tfrac{1}{2}.\mathrm{I}.\omega^2$$
$$= \tfrac{1}{2}.(\tfrac{1}{2}.\mathrm{M}.r^2.).v^2/r^2$$
$$= \tfrac{1}{4}.\mathrm{M}.v^2,$$

and the total kinetic energy of the disc at the bottom of the slope

$$= \tfrac{1}{2}.\mathrm{M}.v^2 + \tfrac{1}{2}.\mathrm{I}.\omega^2$$
$$= \tfrac{1}{2}.\mathrm{M}.v^2 + \tfrac{1}{4}.\mathrm{M}.v^2$$
$$= \tfrac{3}{4}.\mathrm{M}.v^2.$$

This result is in terms of the linear velocity v, at the bottom of the slope. We can express it in terms of the vertical height, h, through which the disc descends, for the total kinetic energy gained by the disc must be equal to the potential energy lost by it, which is, as we have seen, $\mathrm{M}.g.h.$ units. Therefore the total kinetic

energy of the disc at the bottom of the slope is :—

$$= \tfrac{3}{4}.\mathrm{M}.v^2$$
$$= \mathrm{M}.g.h,$$

so that the **kinetic energy of translation** at the foot of the slope is :—

$$= \tfrac{1}{2}.\mathrm{M}.v^2$$
$$= \tfrac{2}{3}.\mathrm{M}.g.h$$

and the **kinetic energy of rotation** at the foot of the slope is :—

$$= \tfrac{1}{4}.\mathrm{M}.v^2$$
$$= \tfrac{1}{3}.\mathrm{M}.g.h.$$

We can now determine the values of the **linear velocity** and angular velocity at the bottom of the slope. We have :—

$$\tfrac{3}{4}.\mathrm{M}.v^2 = \mathrm{M}.g.h,$$

whence $$v^2 = \frac{\mathrm{M}.g.h}{\tfrac{3}{4}.\mathrm{M}}$$
$$= \tfrac{4}{3}.g.h,$$

so that $$v = 2.\sqrt{\frac{gh}{3}} \text{ feet (or cm.) per second.}$$

The linear velocity at the foot of the slope is therefore independent of the *mass* of the disc, or the *radius* of the disc, or the *length* of the slope, and depends only on the vertical height h.

For the **angular velocity**, we have :—

$$\tfrac{1}{2}.\mathrm{I}.\omega^2 = \tfrac{1}{3}.\mathrm{M}.g.h,$$

whence $$\omega^2 = \frac{\tfrac{1}{3}.\mathrm{M}.g.h}{\tfrac{1}{2}.\mathrm{I}}$$
$$= \frac{\tfrac{1}{3}.\mathrm{M}.g.h}{\tfrac{1}{2}(\tfrac{1}{2}\mathrm{M}r^2)}$$
$$= \frac{4.g.h}{3.r^2},$$

so that $$\omega = \frac{2}{r}.\sqrt{\frac{gh}{3}} \text{ radians per second.}$$

The **time occupied** in rolling down the slope is given by :—

$$t = \frac{\text{length of slope}}{\text{mean velocity}}$$

$$= \frac{s}{\frac{1}{2}.v}$$

$$= s.\sqrt{\frac{3}{gh}} \text{ seconds.}$$

The **linear acceleration** of the disc along the plane is given by :—

$$a = \frac{\text{final velocity}}{\text{time occupied}}$$

$$= \frac{v}{t}$$

$$= \frac{2.g.h}{3.s}$$

$$= \tfrac{2}{3}.g.\sin\theta \text{ feet (or cm.) per second per second,}$$

where θ is the angle of inclination of the plane to the horizontal.

DISC ON AXLE ROLLING DOWN INCLINED PLANE. In this case the disc itself does not roll on the inclined plane, but is supported by an axle, which is rigidly attached to it and rolls on two inclined rails or similar supports. We will again assume that there is no slipping, so that the motion is one of rolling without any sliding.

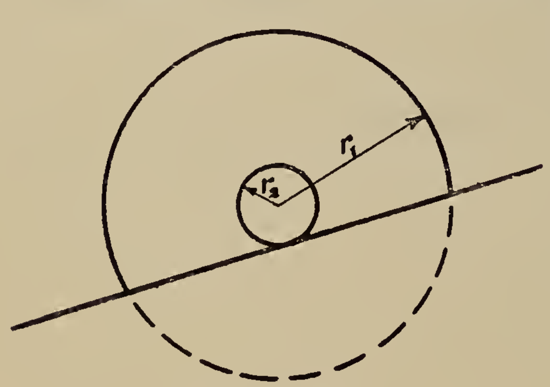


FIG. 118.

Let r_1 = radius of disc,
and r_2 = radius of axle.

Then we have :—

Total kinetic energy at foot of slope, for disc alone, without axle

$$= \tfrac{1}{2}.\mathrm{M}.v^2 + \tfrac{1}{2}.\mathrm{I}.\omega^2$$
$$= \tfrac{1}{2}.\mathrm{M}.v^2 + \tfrac{1}{2}(\tfrac{1}{2}.\mathrm{M}.r_1^2)v^2/r_2^2$$
$$= \tfrac{1}{2}.\mathrm{M}.v^2\left(1 + \frac{r_1^2}{2.r_2^2}\right).$$

If $r_2 = r_1$, then the total kinetic energy

$$= \tfrac{1}{2}.\mathrm{M}.v^2(1 + \tfrac{1}{2})$$
$$= \tfrac{3}{4}.\mathrm{M}.v^2 \quad \text{as before.}$$

The kinetic energy of the *axle* alone will be the kinetic energy of a disc or cylinder rolling down an inclined plane, as already determined, i.e., it will be :—

$$= \tfrac{3}{4}.m.v^2,$$

where m is the mass of the axle. Therefore the total kinetic energy of disc and axle at the bottom of slope is :—

$$= \tfrac{1}{2}.\mathrm{M}.v^2\left(1 + \frac{r_1^2}{2.r_2^2}\right) + \tfrac{3}{4}.m.v^2.$$

The same result could be obtained by direct calculation, noting that the moment of inertia of the disc and axle together is the arithmetical sum of their separate moments of inertia, the axis of rotation being the same in each case.

Total kinetic energy at foot of slope

$$= \tfrac{1}{2}(\mathrm{M} + m)v^2 + \tfrac{1}{2}(\tfrac{1}{2}\mathrm{M}r_1^2 + \tfrac{1}{2}mr_2^2)\frac{v^2}{r_2^2}$$
$$= \tfrac{1}{2}v^2\{\mathrm{M} + m + \tfrac{1}{2}(\mathrm{M}r_1^2/r_2^2) + \tfrac{1}{2}m\}$$
$$= \tfrac{1}{2}v^2\{\mathrm{M}(1 + r_1^2/2r_2^2) + 1\tfrac{1}{2}m\}$$
$$= \tfrac{1}{2}\mathrm{M}v^2(1 + r_1^2/2r_2^2) + \tfrac{3}{4}mv^2 \quad \text{as before.}$$

SUMMARY OF CHAPTER 13

Kinematics deals with the properties of motion itself without considering the forces involved. **Kinetics** deals with the forces concerned in motion.

The **angular momentum** (or **moment of momentum**) of a

rotating body is equal to the product of the moment of inertia of the body about the axis of rotation, and the angular velocity of the body.

The **moment of inertia** (or **second moment of mass**) of a body, about any given axis, is the sum of the products of the mass of each particle of the body into the square of its distance from that axis.

The **radius of gyration** of a body, about any given axis, is the effective distance from that axis at which we may consider the whole mass of the body to be concentrated. The moment of inertia of a body about any given axis is equal to the product of the mass of the body and the square of its radius of gyration about the same axis.

The systematic absolute units of moment of inertia are **one pound-foot-square,** and **one gramme-centimetre-square.** The dimensions of moment of inertia are $[M][L]^2$. The dimensions of angular momentum are $[M][L]^2[T]^{-1}$.

The **rate of change of angular momentum** of a body is equal to the product of the moment of inertia of the body, about its axis of rotation, and its angular acceleration.

First Law of Angular Momentum. There is no change in the angular momentum of a body unless the body is acted upon by some external torque.

Second Law of Angular Momentum. When there is a change of angular momentum, then the torque producing it is proportional to the rate of change of angular momentum, and acts in the same direction as the change of angular momentum.

The first of these laws is also known as the **Principle of the Conservation of Angular Momentum.**

Torque may be defined as that which produces, or tends to produce, a change of angular momentum in the body upon which it acts.

Force $=$ mass $\times$ linear acceleration.

Torque $=$ moment of inertia $\times$ angular acceleration.

Angular momentum and torque (or rate of change of angular momentum) are vector quantities of the second class.

The **kinetic energy of rotation** of a body is equal to half the product of the moment of inertia of the body about the axis of rotation and the square of the angular velocity of the body.

The dimensions of kinetic energy of rotation are the same as those of other forms of energy and work.

The work done on a rotating body is equal to the product of the torque acting and the angle turned through.

Kinetic energy of rotation $=$ torque $\times$ angle.

Angular momentum $=$ torque $\times$ time.

The kinetic energy of a solid disc or cylinder rotating about its own axis is equal to $\frac{1}{4}Mv^2$, where M is the mass of the body and v is the linear speed of any point on the circumference.

EXAMPLES XIII

(*For Hints on Working Examples, see page* 21.)

1. *The mass of a rotating body is* 56 *pounds, its radius of gyration is* 15 *inches, and its angular velocity is* 240 *revolutions per minute. Find its moment of inertia, moment of momentum, and kinetic energy.*

Moment of inertia

$= \text{mass} \times \text{square of radius of gyration}$

$= 56 \text{ pounds} \times (1\tfrac{1}{4} \text{ feet})^2$

$= 87\tfrac{1}{2} \text{ pound-feet-square.}$

Moment of momentum

$= \text{moment of inertia} \times \text{angular velocity}$

$= 87\tfrac{1}{2} \text{ pound-feet-square} \times \left(\dfrac{240 \times 2\pi}{60}\right) \text{ radians per second}$

$= 87\tfrac{1}{2} \text{ pound-feet-square} \times 8\pi \text{ radians per second}$

$= 2{,}200 \text{ F.P.S. units.}$

Kinetic energy of rotation

$= \tfrac{1}{2} \text{ moment of inertia} \times \text{square of angular velocity}$

$= \tfrac{1}{2} \times 87\tfrac{1}{2} \text{ pound-feet-square} \times (8\pi \text{ radians per second})^2$

$= \tfrac{1}{2} \times 87\tfrac{1}{2} \times 64 \times 9{\cdot}87$

$= 27{,}650 \text{ foot-poundals}$

$= 858 \text{ foot-pounds.}$

2. *A fly-wheel, consisting of a solid disc of cast-iron* 18 *inches in diameter, and* 4 *inches thick, is to be run at a maximum rim speed of* 100 *feet per second. What is the greatest amount of energy that can be stored in the wheel? What constant force must be applied to the rim of the wheel in order to bring it to rest in* 5 *minutes? The density of cast-iron is* ·26 *pound per cubic inch.*

Moment of inertia of fly-wheel

$= \tfrac{1}{2}.\text{mass in pounds} \times \text{square of radius in feet}$

$= \tfrac{1}{2} \times \tfrac{1}{3}.\tfrac{1}{4}\pi\,(18)^2 \times {\cdot}26 \text{ pounds} \times (\tfrac{3}{4})^2 \text{ feet}^2$

$= \tfrac{1}{2} \times (27\pi \times {\cdot}26) \text{ pounds} \times \tfrac{9}{16} \text{ feet}^2$

$= \tfrac{1}{2} \times 220{\cdot}7 \text{ pounds} \times \tfrac{9}{16} \text{ feet}^2$

$= 62{\cdot}1 \text{ pound-feet-square.}$

Maximum angular velocity

$$= \frac{\text{rim speed}}{\text{radius}}$$

$$= \frac{100 \text{ feet per second}}{\frac{3}{4} \text{ foot}}$$

$$= \frac{400}{3} \text{ radians per second.}$$

Maximum kinetic energy

$$= \tfrac{1}{2}.(\text{moment of inertia}) \times (\text{angular velocity})^2$$

$$= \tfrac{1}{2} \times 62{\cdot}1 \text{ pound-feet}^2 \times \frac{400}{3} \times \frac{400}{3}$$

$$= 552{,}000 \text{ foot-poundals.}$$

Angular momentum

$$= \text{moment of inertia} \times \text{angular velocity}$$

$$= 62{\cdot}1 \text{ pound-feet-square} \times \frac{400}{3} \text{ radians per second}$$

$$= 8{,}280 \text{ F.P.S. units.}$$

Rate of change of angular momentum

$$= \frac{\text{angular momentum destroyed}}{\text{time occupied}}$$

$$= \frac{8{,}280 \text{ F.P.S. units}}{5 \times 60 \text{ seconds}}$$

$$= 27{\cdot}6 \text{ F.P.S. units per second}$$

Torque applied to rim

$$= \text{braking force} \times \text{radius of fly-wheel}$$

$$= \text{P poundals} \times \tfrac{3}{4} \text{ foot.}$$

This must be equal to the rate of change of angular momentum, i.e.,

$$\tfrac{3}{4}\text{P} = 27{\cdot}6 \text{ F.P.S. units per second,}$$

whence

$$\text{P} = 27{\cdot}6 \times \frac{4}{3}$$

$$= 36{\cdot}8 \text{ poundals}$$

$$= 1{\cdot}14 \text{ pounds-weight.}$$

3. *Starting from rest a solid steel cylinder,* 3 *inches in diameter and of* 15 *pounds mass, rolls, without slipping, down a plane* 6 *feet in length, inclined at an angle of* 13 *degrees to the horizontal. Find the linear velocity of the cylinder, its total kinetic energy at the foot of the slope, and the time taken to roll down. Determine the ratio of the kinetic energy of rotation to the kinetic energy of translation.*

Total kinetic energy at foot of slope

$$= \text{potential energy lost}$$
$$= \text{weight of cylinder} \times \text{vertical drop}$$
$$= (15g) \text{ poundals} \times 6 \text{ feet} \times \sin 13^\circ$$

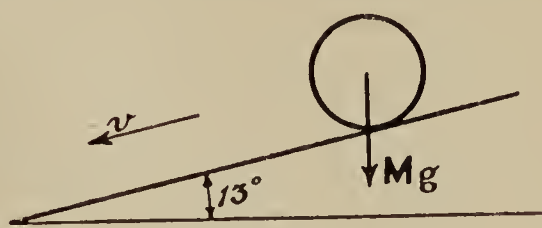


FIG. 119.

$$= (15 \times 32{\cdot}2) \text{ poundals} \times (6 \times {\cdot}225) \text{ feet}$$
$$= 483 \text{ poundals} \times 1{\cdot}35 \text{ feet}$$
$$= 652 \text{ foot-poundals.}$$

Also, total kinetic energy at foot of slope

$$= \text{K.E. of translation} + \text{K.E. of rotation}$$
$$= \tfrac{1}{2}.\text{M}.v^2 + \tfrac{1}{2}.\text{I}.\omega^2$$

(where M = mass in pounds
v = velocity in feet per second
I = mom. of inertia in pound-feet^2
= $\frac{1}{2}(\text{M}.r^2)$ pound-feet^2
r = radius of cylinder in feet
ω = ang. veloc. in rads. per second)

$$= \tfrac{1}{2}.\text{M}.v^2 + \tfrac{1}{2}(\tfrac{1}{2}.\text{M}.r^2)\frac{v^2}{r^2}$$
$$= \tfrac{1}{2}.\text{M}.v^2 + \tfrac{1}{4}.\text{M}.v^2$$
$$= \tfrac{3}{4}.\text{M}.v^2 \text{ foot-poundals.}$$

Therefore ratio of kinetic energy of rotation to kinetic energy of translation

$$= \frac{\frac{1}{4}.\text{M}.v^2}{\frac{1}{2}.\text{M}.v^2}$$
$$= \tfrac{1}{2}.$$

Total kinetic energy at foot of slope

$$= \tfrac{3}{4}.\text{M}.v^2 = 652 \text{ foot-poundals,}$$

whence $$v^2 = \frac{652 \text{ foot-poundals}}{\frac{3}{4}.15 \text{ pounds}}$$

and $$v = \sqrt{\frac{652 \times 4}{3 \times 15}},$$

i.e., velocity at foot of slope

$$= 7{\cdot}61 \text{ feet per second.}$$

Time occupied in rolling down

$$= \frac{\text{length of plane}}{\text{mean velocity}}$$

$$= \frac{6 \text{ feet}}{\frac{1}{2} \times 7 \cdot 61 \text{ feet per second}}$$

$$= 1 \cdot 58 \text{ seconds.}$$

4. *A cast-iron fly-wheel is to be made 3 feet in diameter and of uniform thickness. If the mean speed of the periphery of the wheel is to be a mile a minute, and is not to vary more than* 10 *per cent. for a change of energy in the wheel of 7 foot-tons, find what thickness the wheel must be made.*

Let t = required thickness of wheel in inches.

Moment of inertia of wheel

$$= \tfrac{1}{2}.\text{mass in pounds} \times \text{square of radius in feet}$$

$$= \tfrac{1}{2}\left\{\frac{\pi}{4}(36 \text{ inches})^2 \times t \text{ inches} \times \cdot 26 \text{ pounds per cubic inch} \times (1\tfrac{1}{2} \text{ feet})^2\right\}$$

$$= \tfrac{1}{2} \times 264 \cdot 5t \text{ pounds} \times 2\tfrac{1}{4} \text{ feet}^2$$

$$= 297 \cdot 5t \text{ pound-feet}^2.$$

Mean angular velocity of wheel

$$= \frac{\text{mean peripheral speed}}{\text{radius}}$$

$$= \frac{88 \text{ feet per second}}{1\tfrac{1}{2} \text{ feet}}$$

$$= 58 \cdot 67 \text{ radians per second.}$$

∴ Minimum angular velocity of wheel

$$= 58 \cdot 67 \text{ radians per second} - 10 \text{ per cent.}$$

$$= 58 \cdot 67 - 5 \cdot 867$$

$$= 52 \cdot 80 \text{ radians per second.}$$

Kinetic energy of wheel at mean velocity

$$= \tfrac{1}{2}.\text{moment of inertia} \times \text{square of angular velocity}$$

$$= \tfrac{1}{2}.297 \cdot 5.t \times (58 \cdot 67)^2$$

$$= \tfrac{1}{2}.297 \cdot 5.t \times 3{,}445$$

$$= 512{,}500.t \text{ foot-poundals}$$

$$= 15{,}910.t \text{ foot-pounds.}$$

Kinetic energy of wheel at minimum velocity

$$= \tfrac{1}{2}.297 \cdot 5.t \times (52 \cdot 80)^2$$

$$= \tfrac{1}{2}.297 \cdot 5.t \times 2{,}788$$

$$= 415{,}000.t \text{ foot-poundals}$$

$$= 12{,}890.t \text{ foot-pounds.}$$

$\therefore$ Change of energy for maximum reduction of speed

$$= 15{,}910.t \text{ foot-pounds} - 12{,}890.t \text{ foot-pounds}$$
$$= 3{,}020.t \text{ foot-pounds.}$$

This must be equal to 7 foot-tons.

$$\therefore \qquad 3{,}020.t = 7 \times 2{,}240 \text{ foot-pounds,}$$

whence
$$t = \frac{7 \times 2{,}240}{3{,}020} \text{ inches}$$
$$= 5{\cdot}15 \text{ inches,}$$

i.e., the wheel must be 5·15 inches in thickness.

5. A wheel, of $1\frac{1}{4}$ tons mass and 2 feet 9 inches radius of gyration, revolves 180 times per minute. Determine its moment of inertia, angular momentum, and kinetic energy.

6. The mass of a rotating body is 74 pounds, its angular momentum is 3,000 F.P.S. units, and its kinetic energy is 1,200 foot-pounds. Find the moment of inertia and radius of gyration of the body, and the number of revolutions it makes per minute.

7. The angular momentum of a body is 488 F.P.S. units, its kinetic energy is 3,965 F.P.S. units, and its radius of gyration is ·8 F.P.S. unit. Find the speed of rotation, moment of inertia, and mass of the body, in F.P.S. units.

8. A body, of $\frac{1}{4}$-ton mass, rotates about an axis from which its mean distance is 3 feet 6 inches. If its speed increases from 180 revolutions per minute to 220 revolutions per minute in $1\frac{1}{2}$ minutes, determine the torque acting upon the body and the change of kinetic energy produced.

9. The moment of inertia of a body about its axis of rotation is 2,345 pound-feet-square, and its mass is 3 cwt. What is its radius of gyration? At what speed is it rotating when the amount of work stored in it is 30 foot-tons?

10. The mass of a rotating body is 576 pounds and its mean distance from the axis of rotation is 44 inches. If it is acted upon by a constant torque of 50 pound-feet, find its angular momentum and kinetic energy after $2\frac{1}{4}$ minutes from rest.

11. A fly-wheel, of 2 tons mass and 3 feet radius of gyration, is rotating at 260 revolutions per minute. If the outside diameter of the wheel is 7 feet, find the frictional force which must be exerted by a brake applied to the rim, in order to stop the wheel in $1\frac{1}{2}$ minutes.

12. The maximum rim speed of a fly-wheel is to be 110 feet per second, and the wheel consists of a solid cast-iron disc 3 feet in diameter and 6 inches in thickness. Determine the maximum amount of energy

that can be stored in the wheel. One cubic foot of cast-iron weighs 450 pounds.

13. In the previous question, the wheel, when running at maximum speed, is brought to rest in 2 minutes by the pressure of brakes on the rim. If the coefficient of friction between the brakes and the rim is ·34, find the normal pressure between them.

14. A solid steel cylinder, one foot in diameter and of 250 pounds mass, rolls, without slipping, down a plane 7 feet 6 inches in length inclined at an angle of $8\frac{1}{2}$ degrees to the horizontal. Find its linear and angular momenta at the foot of the slope.

15. In the previous question, determine the magnitude of the force causing translation, and of the torque causing rotation.

16. The moment of inertia of a body about its axis of rotation is 27 million gramme-centimetres-square, and its radius of gyration is 38 centimetres. Determine the mass of the body. What is its angular momentum when revolving at a speed of 150 revolutions per minute ?

17. The mass of a rotating body is 30 kilogrammes and its mean effective distance from the axis of rotation is half a metre. Find the torque which must be applied to it in order that it may receive 1,200 joules of energy in 15 seconds. Determine also the angular velocity which it will then have.

18. A force of 56 pounds-weight is applied tangentially to the rim of a wheel, 3 feet in diameter, for a period of $1\frac{1}{4}$ minutes. If the moment of inertia of the wheel is 2,000 pound-feet-square, find the kinetic energy and angular velocity imparted to it.

19. A pulley, 4 feet 8 inches in diameter, turns through 12 revolutions under the action of a force of 46 pounds-weight applied tangentially to the rim during the whole period. Find the work done.

20. A torque of 50 pound-feet acts on a fly-wheel for 55 seconds, causing it to make 100 revolutions in that time. Find the moment of inertia of the wheel.

21. A disc, of 16 pounds mass and 7 inches in diameter, is mounted on an axle of $\frac{1}{2}$-pound mass and one inch in diameter. If the axle is placed on a pair of paral el rails inclined at an angle of 15 degrees to the horizontal, and the disc and axle allowed to roll down the slcpe 5 feet from rest, without slipping, find the total kinetic energy at the foot of the slope. What is the ratio of the rotational to the translational energy ?

CHAPTER 14: MOMENT OF INERTIA

IN the previous chapter we became better acquainted with the quantity (introduced in Chapter 3) which we usually term the **moment of inertia** of a body. In the present chapter we shall consider the methods generally employed for the determination of the value of this quantity in particular cases, and shall use them to obtain the value in a number of instances.

These methods in general involve the use of the integral calculus: therefore students who are as yet unacquainted with the latter, may find it better to omit this chapter for the present and pass straight on to Chapter 15.

DEFINITIONS. We have seen that we may define the moment of inertia of a body about a given axis as *the sum of the products of the mass of each particle of the body into the square of its distance from the given axis:* that is:—

$$I = \int mx^2,$$

where I is the moment of inertia of the body about the given axis, m is the mass of any particle of the body, and x is the distance of that particle from the given axis.

We have also seen that we can use the term **second moment of mass** instead of moment of inertia, both terms denoting the same quantity, and that an alternative definition is as follows:—*The second moment of mass of a body about any given axis is equal to the product of the mass of the body and the square of its effective distance from that axis*: that is:—

$$I = Mk^2,$$

where M is the mass of the whole body, and k is the effective distance from the given axis. The distance k

is known as the **radius of gyration** of the body about that axis.

MEASUREMENTS AND DIMENSIONS. We may remind ourselves that we measure second moments of mass (that is, moments of inertia) in **pound-feet-square** if we are using the F.P.S. system, and in **gramme-centimetres-square** if we are using the C.G.S. system. Radii of gyration are of course measured in the usual units of length, that is, feet or centimetres.

We may also recall that the dimensions of moment of inertia are $[M][L]^2$, and those of radius of gyration are $[L]$.

It is not very easy to realise the moment of inertia of a body as a definite physical quantity, and it is perhaps better therefore, at first, just to consider it as a convenient term that we employ for the quantity $\int mx^2$, because of the frequent occurrence in our calculations of the latter quantity.

CALCULATION OF MOMENTS OF INERTIA. It should be realised that any given body has more than one moment of inertia: it has, in fact, an infinite number, for we can consider the body with regard to any number of different axes, and the body has a moment of inertia about every such axis. If, however, both the body and the axis are definitely specified, then there is only one moment of inertia for that case.

The **form** of the expression for the moment of inertia of a particular body of uniform density depends only upon the **shape** of the body and the **position of the axis** with regard to the body: that is, it depends upon the *distribution* of the mass with regard to the axis.

The **numerical value** of the moment of inertia depends also upon the **size** and **density** of the body: that is, it depends upon the *nature* and *quantity* of the material of the body, as well as its distribution.

In the case of simple symmetrical bodies, we can usually find an expression for the moment of inertia about certain axes with great ease by means of the integral calculus: fortunately these simple cases are

also the most important. The calculation of numerical values is then merely a matter of substitution.

The first step, in the determination of the moment of inertia of a body, is to find the **mass of an element** of the body, choosing the element in such a way that we can express its mass in terms of geometry of the body, and of its density. Such an element, being indefinitely small, may be considered as a particle of the body: therefore, *having found an expression for the mass of the element, we substitute it for the term m in the general expression* $\int mx^2$, *and integrate*. This gives us an expression for the moment of inertia of the whole body in terms of the geometrical measurements of the body: the *limits* between which we integrate will depend upon the position of the axis with regard to the body. Then, if we know the numerical values of the geometrical measurements of the body, we can substitute these and so obtain the numerical value of the moment of inertia.

The method will be more readily understood when we have applied it to certain definite cases: we will therefore apply it forthwith. In considering these examples the student should endeavour not merely to understand the details of the particular instances taken, but also to fix as clearly as possible in his mind the *principles* involved. If this is done, then no difficulty will be found in adapting the methods to suit other cases.

1. Solid Cylinder or Disc, about its own axis.

Consider an elementary ring, of radius x, breadth dx, and thickness b (parallel to the axis).

The *volume* of this elementary ring is :—

$$v = 2\pi x.dx.b$$

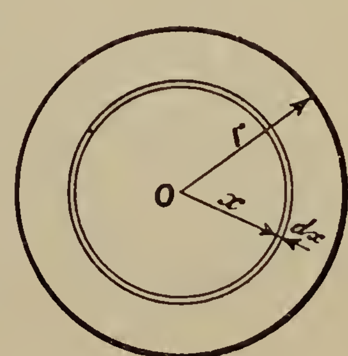


FIG. 120.

and its *mass* is :—

$$m = \varrho.v, \text{ where } \varrho \text{ (rho) is the density of the material.}$$
$$= \varrho.2\pi x.dx.b$$
$$= 2\pi b\varrho.x.dx.$$

The *moment of inertia* of this elementary ring (every particle of which is at the same distance x from the given axis) is :—

$$i = m.x^2$$
$$= 2\pi b\varrho.x^3.dx.$$

Therefore the moment of inertia of the whole body about the given axis is :—

$$\begin{aligned} \mathrm{I} &= \int i \\ &= \int_o^r m.x^2 \\ &= \int_o^r 2\pi b\varrho.x.^3dx \\ &= 2\pi b\varrho \int_o^r x^3.dx \\ &= 2\pi b\varrho \left[\tfrac{1}{4}x^4\right]_o^r \\ &= 2\pi b\varrho.\tfrac{1}{4}r^4 \\ &= \tfrac{1}{2}(\pi r^2.b\varrho).r^2 \\ &= \tfrac{1}{2}.\mathrm{M}.r^2, \end{aligned}$$

where M is the mass of the whole cylinder or disc, and r is its radius. If M is in pounds and r in feet, then I will be in *pound-feet-square*. If C.G.S. units are employed, then I will be in *gramme-centimetres-square*.

We have seen already that, for any body, the moment of inertia about any given axis is equal to the product of the mass of the body and the square of its *radius of gyration* about that axis, that is :—

$$\mathrm{I} = \mathrm{M}.k^2.$$

In the particular case which we are now considering we have found that $\mathrm{I} = \frac{1}{2}.\mathrm{M}.r^2$. Therefore we have :—

$$\mathrm{M}.k^2 = \tfrac{1}{2}.\mathrm{M}.r^2,$$

whence $$k^2 = \tfrac{1}{2}.r^2,$$

so that $$k = \frac{r}{\sqrt{2}}$$
$$= \cdot 707\, r \text{ (feet or cm.)},$$

i.e., the *radius of gyration* of a solid cylinder or disc about its own axis, is equal to ·707 of the *radius* of the cylinder or disc.

2. Hollow Cylinder or Disc, about its own axis.

The moment of inertia in this case is found in exactly the same way as for a solid cylinder, but the limits of integration are now r_1, the outside radius, and r_2, the inside radius. We have therefore :—

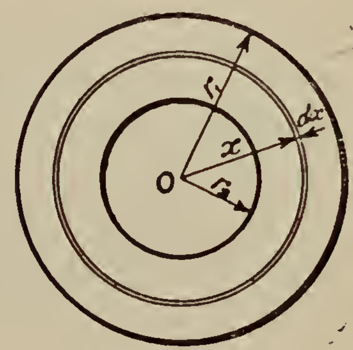


FIG. 121.

$$\begin{aligned} I &= 2\pi b\varrho \int_{r_2}^{r_1} x^3.dx \\ &= 2\pi b\varrho \Big[\tfrac{1}{4}x^4 \Big]_{r_2}^{r_1} \\ &= 2\pi b\varrho.\tfrac{1}{4}(r_1{}^4 - r_2{}^4) \\ &= \tfrac{1}{2}.\pi b\varrho(r_1{}^2 - r_2{}^2).(r_1{}^2 + r_2{}^2) \\ &= \tfrac{1}{2}.M.(r_1{}^2 + r_2{}^2), \end{aligned}$$

where M is the mass of the whole cylinder or disc.

If $r_2 = 0$, then the cylinder is solid, and we have
$$I = \tfrac{1}{2}.M.r_1{}^2 \quad \text{as before.}$$

To determine the radius of gyration, we have :—
$$M.k^2 = \tfrac{1}{2}.M.(r_1{}^2 + r_2{}^2),$$
whence $$k^2 = \tfrac{1}{2}.(r_1{}^2 + r_2{}^2)$$
and $$k = \sqrt{\tfrac{1}{2}(r_1{}^2 + r_2{}^2)}.$$

When the inside radius, r_2, = 0, the cylinder is solid, and we have :—
$$k = \sqrt{\tfrac{1}{2}(r_1{}^2 + 0)}$$
$$= r_1/\sqrt{2} \quad \text{as before.}$$

When the inside radius is very nearly equal to the

outside radius, the cylinder is extremely thin, and we have, approximately :—

$$r_1 = r_2 = r \quad \text{say.}$$

Then
$$k = \sqrt{\tfrac{1}{2}(r^2 + r^2)}$$
$$= \sqrt{r^2}$$
$$= r$$

as we should expect, for the whole of the mass is now at practically the same distance from the axis.

3. Thin Rod, about axis through its centre of mass perpendicular to its length.

Let l = length of rod in feet (or cm.),
and a = its cross-sectional area in sq. ft. (or sq. cm.),
ϱ = the density of the material.

Then the mass of an element of the rod, of length dx, is :—

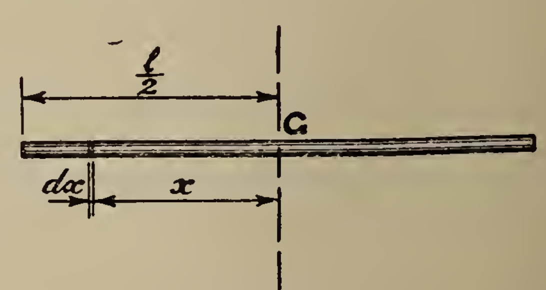


FIG. 122.

$$m = a.dx.\varrho,$$

so that
$$i = m.x^2$$
$$= a.\varrho.x^2.dx$$

and
$$\mathrm{I} = \int i$$
$$= \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} m.x^2$$
$$= \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} a.\varrho.x^2.dx$$
$$= a.\varrho \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} x^2.dx$$
$$= a.\varrho \left[\tfrac{1}{3}x^3\right]_{-\frac{1}{2}l}^{+\frac{1}{2}l}$$
$$= a.\varrho.\tfrac{1}{3}\left(\frac{l^3}{8} + \frac{l^3}{8}\right)$$

$$= \tfrac{1}{12}(a.\varrho.l)l^2$$
$$= \tfrac{1}{12}.M.l^2,$$

where M is the mass of the whole rod.

To determine the radius of gyration, we have :—

$$M.k^2 = \tfrac{1}{12}.M.l^2,$$

whence $$k^2 = \tfrac{1}{12}l^2$$

and $$k = \frac{l}{2\sqrt{3}}.$$

It must be noted that these results are *strictly* true only when the sectional area, a, is *indefinitely* small: they will, however, be *approximately* true if a is *relatively* small in comparison with l.

4. Thin Rod, about axis through one end perpendicular to length.

This case closely resembles the previous one, the only difference being that x now varies from 0 to l, and the limits of integration are correspondingly altered.

$$I = a.\varrho\int_0^l x^2.dx$$
$$= a.\varrho\left[\tfrac{1}{3}x^3\right]_0^l$$
$$= \tfrac{1}{3}(a.\varrho.l)l^2$$
$$= \tfrac{1}{3}.M.l^2,$$

The radius of gyration is given by :—

$$k^2 = \tfrac{1}{3}.l^2,$$

whence $$k = \frac{l}{\sqrt{3}}.$$

5. Rectangular Lamina, about one edge.

In this case we take for our element of mass the mass of a narrow strip of the lamina parallel with the edge about which we require the moment of inertia.

Let b = breadth of lamina,
d = depth of lamina,
t = thickness of lamina,
and ϱ = density of the material.

Then, mass of elementary strip :—

$$= m$$
$$= b.t.dx.\varrho.$$

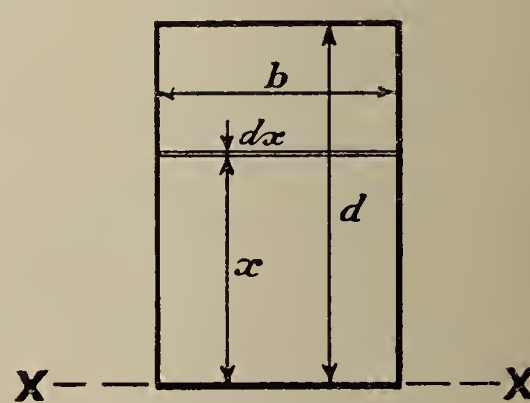


FIG. 123.

Moment of inertia of strip about given axis :—

$$= i = m.x^2$$
$$= b.t.\varrho.x^2.dx.$$

Moment of inertia of whole lamina about given axis

$$= \mathrm{I} = \int i$$
$$= \int_o^d m.x^2$$
$$= \int_o^d b.t.\varrho.x^2.dx$$
$$= b.t.\varrho \int_o^d x^2.dx$$
$$= b.t.\varrho \left[\tfrac{1}{3}x^3 \right]_o^d$$
$$= b.t.\varrho.\tfrac{1}{3}.d^3$$
$$= \tfrac{1}{3}(b.t.d.\varrho)d^2$$
$$= \tfrac{1}{3}.\mathrm{M}.d^2.$$

The radius of gyration is given by

$$k^2 = \frac{\mathrm{I}}{\mathrm{M}}$$
$$= \tfrac{1}{3}.d^2,$$

whence $$k = \frac{d}{\sqrt{3}}.$$

6. Rectangular Lamina, about axis through centre of gravity parallel to one edge.

The element of mass will be taken as in the previous case, the only difference between the two cases being in the limits of integration.

$$\mathrm{I} = b.t.\varrho \int_{-\frac{1}{2}d}^{+\frac{1}{2}d} x^2.dx$$
$$= b.t.\varrho \Big[\tfrac{1}{3}x^3\Big]_{-\frac{1}{2}d}^{+\frac{1}{2}d}$$
$$= b.t.\varrho.\tfrac{1}{3}\Big(\frac{d^3}{8} + \frac{d^3}{8}\Big)$$
$$= \tfrac{1}{12}(b.d.t.\varrho).d^2$$
$$= \tfrac{1}{12}.\mathrm{M}.d^2.$$

The radius of gyration, k,

$$= \sqrt{\frac{\mathrm{I}}{\mathrm{M}}}$$

$$= \frac{d}{2\sqrt{3}}.$$

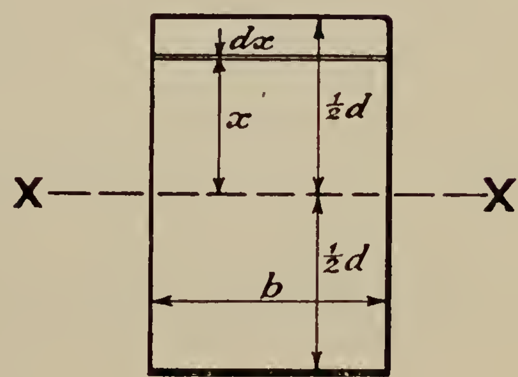


FIG. 124.

It should be noted that the results obtained for rectangles are equally true for parallelograms, as shown below.

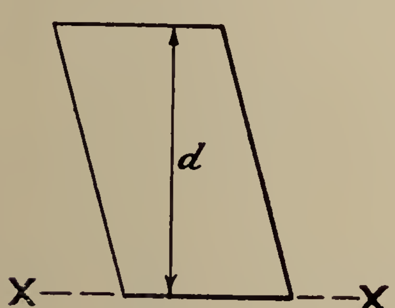


FIG. 125.

$\mathrm{I} = \tfrac{1}{3}.\mathrm{M}.d^2.$

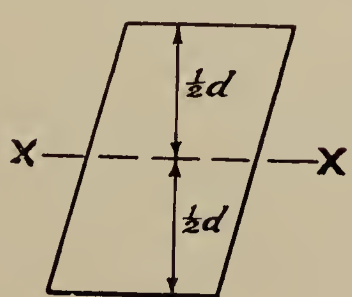


FIG. 126.

$\mathrm{I} = \tfrac{1}{12}.\mathrm{M}.d^2.$

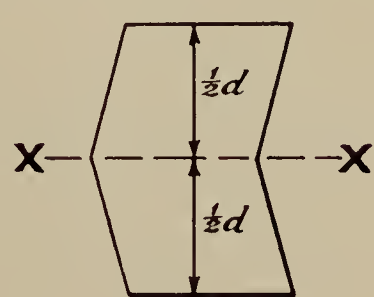


FIG. 127.

$\mathrm{I} = \tfrac{1}{12}.\mathrm{M}.d^2.$

7. Triangular Lamina, about one edge.

Area of elementary strip

$$= b\Big(\frac{h - x}{h}\Big)dx.$$

Mass of elementary strip

$$= b.t.\varrho\left(\frac{h - x}{h}\right)dx.$$

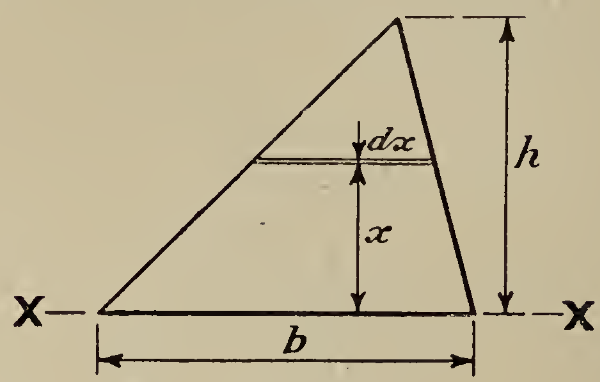


FIG. 128.

Moment of inertia of elementary strip

$$= b.t.\varrho\left(\frac{h - x}{h}\right)x^2.dx.$$

$$= \frac{b.t.\varrho}{h}(hx^2 - x^3)dx.$$

Moment of inertia of whole triangle

$$= \frac{b.t.\varrho}{h}\int_o^h(h.x^2 - x^3)dx$$

$$= \frac{b.t.\varrho}{h}\left[h\frac{x^3}{3} - \frac{x^4}{4}\right]_o^h$$

$$= \frac{b.t.\varrho}{h}h^4(\tfrac{1}{3} - \tfrac{1}{4})$$

$$= \tfrac{1}{12}b.t.\varrho.h^3$$

$$= \tfrac{1}{6}(\tfrac{1}{2}.b.h.t.\varrho)h^2$$

$$= \tfrac{1}{6}.\text{M}.h^2.$$

Radius of gyration $= k$

$$= \sqrt{\frac{\text{I}}{\text{M}}}$$

$$= \sqrt{\tfrac{1}{6}.h^2}$$

$$= \frac{h}{\sqrt{6}}.$$

8. Parallelogrammic Lamina, about a diagonal.

This may be considered as the sum of two equal triangles with their bases in the same straight line, which forms the diagonal XX (Fig. 129).

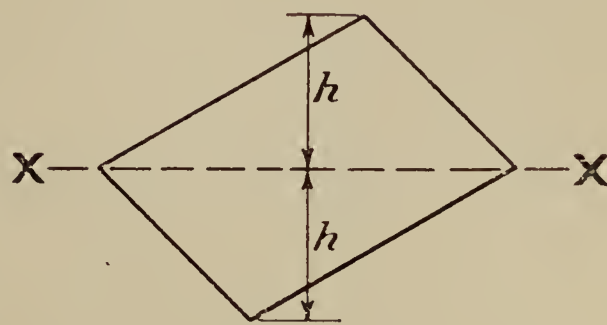


FIG. 129.

Therefore the moment of inertia of the parallelogrammic lamina, about XX, will be equal to the sum of the moments of inertia of the two triangular laminæ about the same line,

i.e., $I = 2 \times \frac{1}{6}.M.h^2$

$= \frac{1}{3}.M.h^2$, where M is the mass of each triangular lamina.

$= \frac{1}{6}.M.h^2$, where M is the mass of the parallelogrammic lamina.

In the particular case where the parallelogram is a square, of side a, then the height h is equal to $\frac{1}{2}a\sqrt{2}$, so that the moment of inertia of a square lamina about a diagonal is

$$\begin{aligned} I &= \tfrac{1}{6}.M.h^2 \\ &= \tfrac{1}{6}.M.(\tfrac{1}{2}a\sqrt{2})^2 \\ &= \tfrac{1}{6}.M.\tfrac{1}{2}.a^2 \\ &= \tfrac{1}{12}.M.a^2. \end{aligned}$$

It should be noted that the thickness of the lamina in any of these results must be *indefinitely* small if the results are to be *strictly* accurate. They will, however, be *sufficiently* accurate for all practical purposes if the thickness is *very* small compared with the other dimensions of the lamina.

Before proceeding further we must consider two theorems dealing with moments of inertia. These theorems are of very great service, for they enable us to find, with great ease, the moment of inertia in many cases which would otherwise present much difficulty. Time spent in making a careful study of them will therefore be well invested.

PERPENDICULAR AXES THEOREM. *The sum of the moments of inertia of any lamina about any two perpendicular axes in its plane, is equal to the moment of inertia of the lamina about an axis perpendicular to the lamina and passing through the point of intersection of the other two axes.*

Let I_X be the moment of inertia of the lamina about the axis XX, I_Y the moment of inertia about the perpendicular axis YY (both XX and YY being in the plane of the lamina), and I_Z the moment of inertia of the lamina about an axis, perpendicular to the lamina, passing through O, the point of intersection of XX and YY (see Fig. 130).

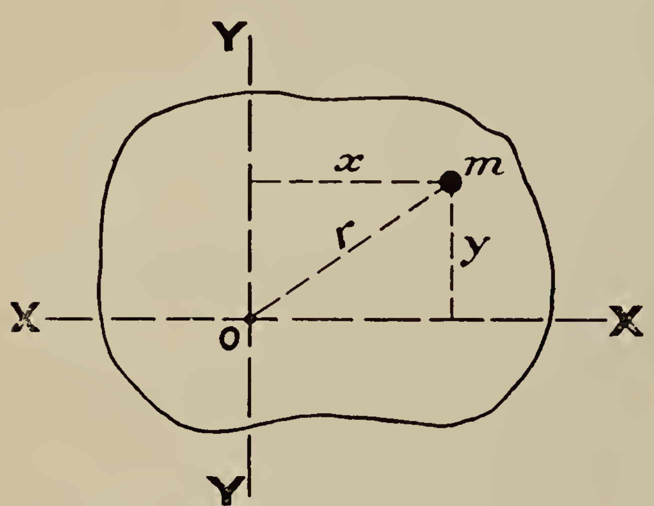


FIG. 130.

Let m be the mass of any particle of the lamina, which is at a distance x from the axis YY, a distance y from the axis XX, and a distance r from the axis ZZ through O. Then :—

$$\begin{aligned} I_Z &= \int m.r^2 \\ &= \int m(x^2 + y^2) \\ &= \int m.x^2 + \int m.y^2 \\ &= I_Y + I_X, \end{aligned}$$

which proves the theorem.

PARALLEL AXES THEOREM. *The moment of inertia of a body about any axis, is equal to the moment of inertia of the body about a parallel axis through its centre of mass, plus the product of the mass of the body and the square of the distance between the two axes :* that is :—

$$I_X = I_G + Ml^2,$$

where I_X = moment of inertia of the body about any axis XX,

I_G = moment of inertia of the body about a parallel axis through the centre of mass of the body,

and l = perpendicular distance between the two axes.

Suppose that m is the mass of any particle of the body, and that the distance of this particle from the given axis XX is x, and its distance from the parallel axis through the centre of mass is y (see Fig. 131).

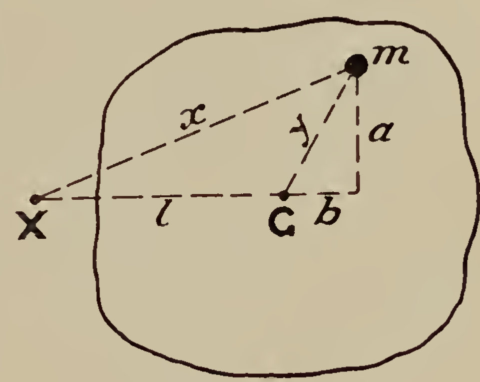


FIG. 131.

Let a be the perpendicular distance of the particle from the plane containing the two axes, and b the distance from the axis through the centre of mass to the foot of the perpendicular from the particle to the plane containing the two axes. A glance at the diagram will make this clear. The two axes are taken as being perpendicular to the plane of the paper, and passing through the points G and X respectively, where G represents the centre of mass of the body.

The moment of inertia of the body, about the axis through X, is :—

$$
\begin{aligned}
I_X &= \int m.x^2 \\
&= \int m.\{a^2 + (b + l)^2\} \\
&= \int m.(a^2 + b^2 + l^2 + 2.b.l) \\
&= \int m.(y^2 + l^2 + 2.b.l) \\
&= \int m.y^2 + \int m.l^2 + 2.l \int m.b \\
&= I_G + M.l^2,
\end{aligned}
$$

for $\int m.b$ is zero, since b is the perpendicular distance of

the particle from the axis through the centre of mass of the body. The theorem is therefore proved.

By means of this theorem we can obtain the moment of inertia of a body about *any* axis, provided that we can obtain the moment of inertia about a parallel axis through the centre of mass of the body. It therefore enables us to find moments of inertia in many cases that would otherwise be very difficult to deal with.

We will now take some further examples of the determination of moments of inertia, involving the use of the two theorems that we have just considered.

9. Circular Lamina, about a diameter.

By the Perpendicular Axes Theorem :—

$$I_X + I_Y = I_Z$$

and by symmetry $\quad I_X = I_Y,$

therefore $\quad I_X = \frac{1}{2}.I_Z$

$$= \frac{1}{2}.(\frac{1}{2}.M.r^2)$$

$$= \frac{1}{4}.M.r^2,$$

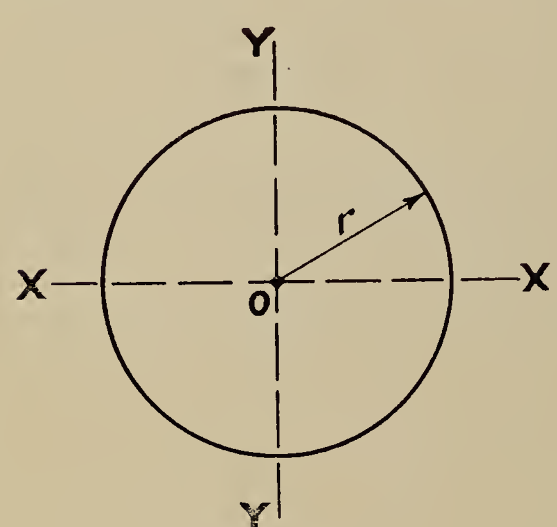


FIG. 132.

where M is the mass of the whole lamina and r is its radius. The value of I_Z is taken from (1) on page 277.

The radius of gyration of a circular lamina about a diameter is therefore :—

$$k = \sqrt{\frac{I}{M}}$$

$$= \sqrt{\tfrac{1}{4}.r^2}$$

$$= \tfrac{1}{2}.r.$$

10. Circular Ring Lamina, about a diameter.

By the same reasoning as in the previous case :—

$$I_X = \tfrac{1}{4}.M.(r_1^2 + r_2^2)$$

and $\quad k = \frac{1}{2}.\sqrt{r_1^2 + r_2^2}.$

The value of I_Z is taken from (2) on page 279.

11. Rectangular Lamina, about a perpendicular axis through its centre of gravity.

By the Perpendicular Axes Theorem :—

$$\begin{aligned} I_Z &= I_X + I_Y \\ &= \tfrac{1}{12}.M.d^2 + \tfrac{1}{12}.M.b^2 \\ &= \tfrac{1}{12}.M.(d^2 + b^2), \end{aligned}$$

so that $$k = \tfrac{1}{2}.\sqrt{\tfrac{1}{3}.(d^2 + b^2)}.$$

The values of I_X and I_Y are taken from (6) on page 283.

In the particular case where $b = d$, the lamina is *square*, and we have :—

$$I_Z = \tfrac{1}{6}.M.d^2$$

and $$k = \frac{d}{\sqrt{6}}.$$

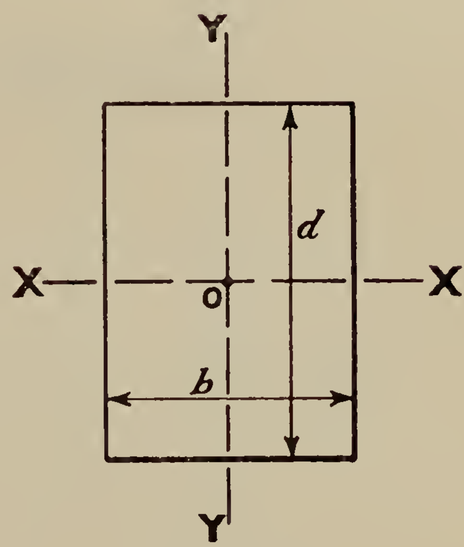


Fig. 133.

12. Solid Cylinder or Disc, about an axis parallel to its own axis.

By the Parallel Axes Theorem :—

$$I_X = I_G + M.l^2,$$

where I_G is the moment of inertia of the disc about its own axis, I_X is the moment of inertia about a parallel axis through any point X, and l is the distance between the two axes,

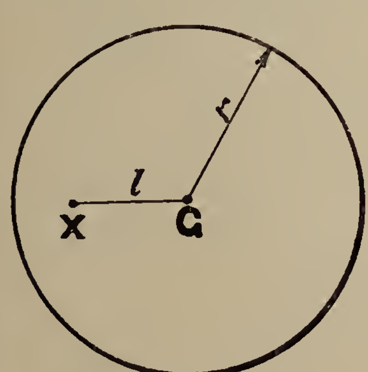


Fig. 134.

i.e., $$\begin{aligned} I_X &= \tfrac{1}{2}.M.r^2 + M.l^2 \\ &= M.(\tfrac{1}{2}r^2 + l^2). \end{aligned}$$

If the point X lies on the rim of the disc, then $l = r$, and we have :—

$$\begin{aligned} I_X &= M.(\tfrac{1}{2}r^2 + r^2) \\ &= 1\tfrac{1}{2}.M.r^2. \end{aligned}$$

13. Rectangular Prism, about a perpendicular axis through its centre of gravity.

The prism may be considered to be composed of an infinite number of laminæ, all of the same size and shape,

and with the axis passing through the centre of gravity of each lamina. Then if l is the length of the prism, and b is its breadth, the moment of inertia of each lamina is, by (11) above, $\frac{1}{12}.M.(l^2 + b^2)$, and the moment of inertia of the whole prism is the sum of the moments of inertia of all the laminæ together, i.e., for whole prism :—

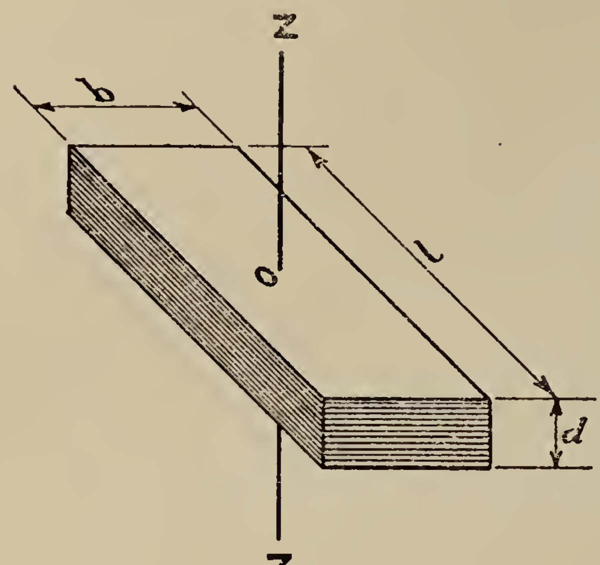


FIG. 135.

$$I = \frac{1}{12}.M.(l^2 + b^2),$$

M being now the mass of the whole prism.

14. Cube, about a diagonal of one face.

Let a = edge of cube,

and m = mass of cube *per unit length.*

We take as element a thin slice of the cube, parallel to the face containing the axis, of thickness dx and hence of mass $m.dx$.

Let i_X = moment of inertia of element of cube about given axis, XX,

and i_O = moment of inertia of element of cube about parallel axis, OO, through centre of mass of *element*.

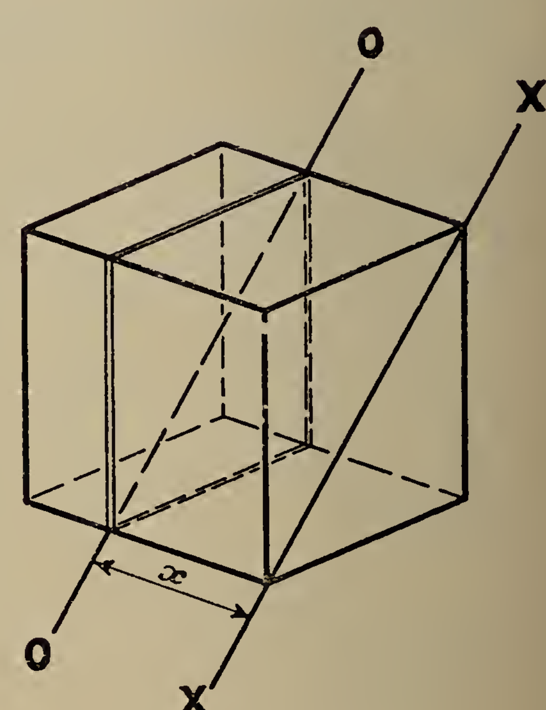


FIG. 136.

By the Parallel Axes Theorem :—

$$i_X = i_O + (m.dx).x^2$$

$$= \frac{1}{12}.(m.dx).a^2. + (m.dx).x^2 \quad \text{(page 285).}$$

Therefore moment of inertia of whole cube about given axis :—

$$= \mathrm{I}_x = \tfrac{1}{12}.m.a^2.\int_o^a dx + m\int_o^a x^2.dx$$

$$= \tfrac{1}{12}.m.a^2.\Big[x\Big]_o^a + m.\Big[\tfrac{1}{3}.x^3\Big]_o^a$$

$$= \tfrac{1}{12}.m.a^3. + m.\tfrac{1}{3}.a^3$$

$$= \tfrac{5}{12}(m.a)a^2$$

$$= \tfrac{5}{12}.\mathrm{M}.a^2,$$

where M is the mass of the whole cube.

The same method could be applied to determine the moment of inertia of the prism in case 13, above. The student should notice that in these cases we apply the Parallel Axes Theorem to the moment of inertia *of the element*: not to the moment of inertia of the body as a whole, as we did in case 12, above. The two cases should be carefully compared.

15. Solid Disc or Cylinder, about a diameter through its centre of mass.

This is another example of the method employed in case 14, above.

Let r = radius of disc or cylinder,

l = its length,

and m = its mass per unit length.

By the Parallel Axes Theorem :—

$$i_x = i_o + (m.dx)x^2$$
$$= \tfrac{1}{4}(m.dx)r^2 + (m.dx)x^2 \text{ (page 288)}$$
$$= \tfrac{1}{4}.m.r^2.dx + m.x^2.dx,$$

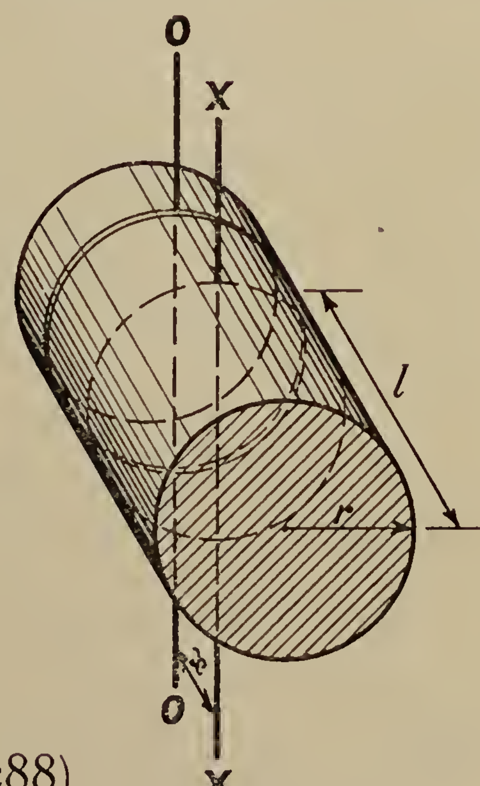


FIG. 137.

whence $$I_X = \tfrac{1}{4}.m.r^2.\int_{-\frac{1}{2}l}^{+\frac{1}{2}l} dx + m\int_{-\frac{1}{2}l}^{+\frac{1}{2}l} x^2.dx$$
$$= \tfrac{1}{4}.m.r^2.\Big[x\Big]_{-\frac{1}{2}l}^{+\frac{1}{2}l} + m\Big[\tfrac{1}{3}.x^3\Big]_{-\frac{1}{2}l}^{+\frac{1}{2}l}$$
$$= \tfrac{1}{4}.m.r^2.l + \tfrac{1}{12}.m.l^3$$
$$= m.l(\tfrac{1}{4}.r^2 + \tfrac{1}{12}.l^2)$$
$$= M.(\tfrac{1}{4}.r^2 + \tfrac{1}{12}.l^2).$$

If r is negligibly small compared with l, then the cylinder is a thin rod, and we have :—
$$I = M.(\tfrac{1}{4}.0 + \tfrac{1}{12}.l^2)$$
$$= \tfrac{1}{12}.M.l^2,$$
which is the result that we have already found in case 3.

16. Solid Disc or Cylinder, about a diameter of one end.

This closely resembles the previous case, the only difference being in the limits of integration. We have, therefore :—
$$I_X = \tfrac{1}{4}.m.r^2.\int_0^l dx + m\int_0^l x^2.dx$$
$$= \tfrac{1}{4}.m.r^2.\Big[x\Big]_0^l + m\Big[\tfrac{1}{3}.x^3\Big]_0^l$$
$$= \tfrac{1}{4}.m.r^2.l + \tfrac{1}{3}.m.l^3$$
$$= m.l.(\tfrac{1}{4}.r^2 + \tfrac{1}{3}.l^2)$$
$$= M.(\tfrac{1}{4}.r^2 + \tfrac{1}{3}\, l^2).$$

If r is negligibly small compared with l, then again the cylinder is a thin rod, and we have :—
$$I = M.(\tfrac{1}{4}.0 + \tfrac{1}{3}.l^2)$$
$$= \tfrac{1}{3}.M.l^2,$$
which is the result that we have already found in case 4.

17. Thin Spherical Shell, about a diameter.

Let m = mass of shell per unit area of surface,
and r = radius of shell.

Also let XX be the axis about which we require the moment of inertia.

Imagine a plane perpendicular to XX and passing through O, the centre of the shell: this plane is represented by YY in Fig. 138. We take as our element a narrow zone, every point of which is at a distance x from the plane through YY, and at a distance $\sqrt{r^2 - x^2}$ from the axis XX.

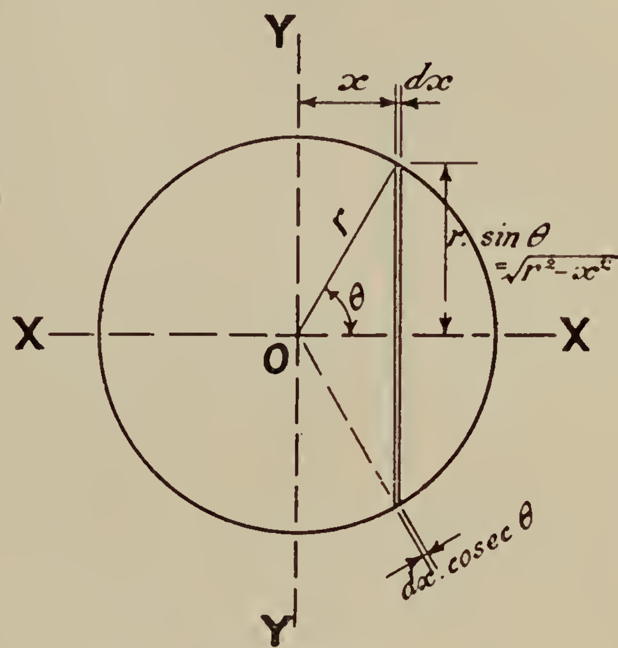


Fig. 138.

Area of this elementary zone

$= (2.\pi.r.\sin\theta)(dx.\text{cosec}\,\theta)$ (See Fig. 138.)

$= 2.\pi.r.dx.$

Mass of elementary zone

$= 2.\pi.m.r.dx.$

Moment of inertia of elementary zone about XX

$= 2.\pi.m.r.dx.(r^2 - x^2).$

Therefore moment of inertia of whole shell about XX

$$= 2.\pi.m.r. \int_{-r}^{+r} (r^2 - x^2)dx.$$

$$= 2.\pi.m.r. \int_{-r}^{+r} r^2.dx - 2.\pi.m.r \int_{-r}^{+r} x^2.dx$$

$$= 2.\pi.m.r.r^2.\Big[x\Big]_{-r}^{+r} - 2.\pi.m.r.\Big[\tfrac{1}{3}.x^3\Big]_{-r}^{+r}$$

$$= 2.\pi.m.r(2.r^3 - \tfrac{2}{3}.r^3)$$

$$= 4.\pi.m.r(r^3 - \tfrac{1}{3}.r^3)$$

$$= \tfrac{2}{3}.(4.\pi.r^2.m).r^2$$

$$= \tfrac{2}{3}.M.r^2,$$

where M is the mass of the whole spherical shell.

The radius of gyration $= k$

$$= \sqrt{\frac{I}{M}}$$

$$= \sqrt{\tfrac{2}{3}.r^2}$$

$$= \sqrt{\tfrac{2}{3}}.r.$$

18. Solid Sphere, about a diameter.

Let r = radius of sphere,
and ϱ = its density.

Also let XX be the axis about which we require the moment of inertia.

Imagine a plane perpendicular to XX and passing through O, the centre of the sphere: this plane is represented by YY in Fig. 139. We take as our element of mass a thin slice of the sphere, forming a disc with its centre on the axis XX and with every point of it at a distance x from the plane through YY.

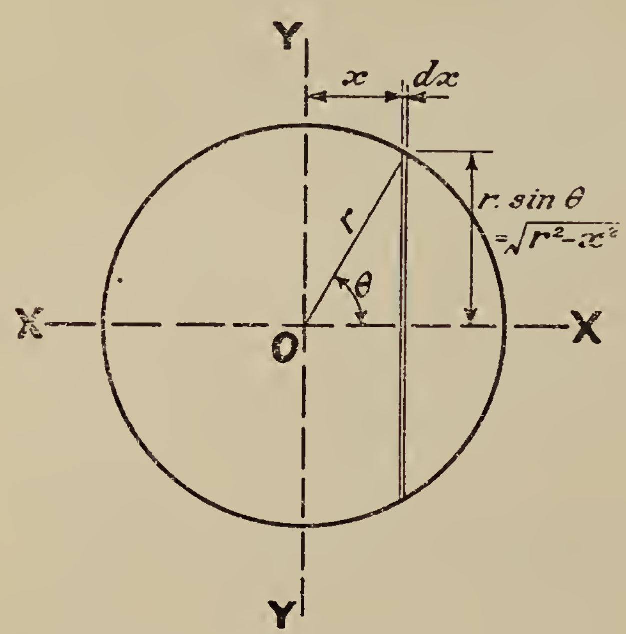


FIG. 139.

Volume of this elementary disc

$$= \pi.(r.\sin\,\theta)^2.dx$$
$$= \pi.(r^2 - x^2).dx.$$

Mass of elementary disc

$$= \pi.\varrho.(r^2 - x^2).dx.$$

Moment of inertia of elementary disc about XX

$$= \pi.\varrho.(r^2 - x^2)dx \times \tfrac{1}{2}(r.\sin\,\theta)^2 \quad \text{(by 1, p. 277)}$$
$$= \tfrac{1}{2}.\pi.\varrho(r^2 - x^2)^2.dx$$
$$= \tfrac{1}{2}.\pi.\varrho(r^4 - 2.r^2.x^2. + x^4).dx.$$

Therefore moment of inertia of whole sphere about XX

$$= \tfrac{1}{2}.\pi.\varrho \int_{-r}^{+r} (r^4 - 2.r^2.x^2. + x^4)dx.$$
$$= \tfrac{1}{2}.\pi.\varrho\{r^4.(2r) - 2.r^2.(\tfrac{2}{3}.r^3) + \tfrac{2}{5}.r^5\}$$
$$= \tfrac{1}{2}.\pi.\varrho.(\tfrac{16}{15}.r^5)$$
$$= \tfrac{8}{15}.\pi.\varrho.r^5$$
$$= \tfrac{2}{5}.(\tfrac{4}{3}.\pi.\varrho.r^3).r^2$$
$$= \tfrac{2}{5}.\text{M}.r^2,$$

where M is the mass of the whole sphere.

The radius of gyration $= k$

$$= \sqrt{\frac{I}{M}}$$

$$= \sqrt{\tfrac{2}{5}.r^2}$$

$$= \sqrt{\tfrac{2}{5}}.r.$$

The same results may be obtained by regarding the sphere as composed of an infinite number of thin concentric shells of radii from 0 to r. Taking x as the variable radius, the moment of inertia of each shell about XX is, by (17) above :—

$$i = \tfrac{8}{3}.\pi.\varrho.x^4.dx.$$

Here $\varrho.dx =$ mass per unit area of shell $= m$ in (17).

Therefore moment of inertia of sphere is

$$\begin{aligned} I &= \tfrac{8}{3}.\pi.\varrho \int_o^r x^4.dx \\ &= \tfrac{8}{3}.\pi.\varrho.\tfrac{1}{5}.r^5 \\ &= \tfrac{8}{15}.\pi.\varrho.r^5 \qquad \text{as before.} \end{aligned}$$

SECOND MOMENT OF AREA. *The* **second moment of an area,** *about a given axis, is the sum of the products of each element of the area into the square of its distance from the given axis* : that is :—

$$I = \int ax^2,$$

where I is the second moment of the area about the given axis, a is any element of the area, and x is the distance of that element from the given axis.

Alternatively we can say that *the second moment of an area about a given axis is equal to the product of the area and the square of its effective distance from that axis* : that is :—

$$I = Ak^2,$$

where A is the whole area, and k is the effective distance from the given axis. The distance k is known as the **radius of gyration** of the area about that axis.

The second moment of an area is sometimes termed the *moment of inertia* of the area : this term is incorrect and should be avoided. An area has no mass and therefore no inertia : consequently it cannot possibly have a moment of inertia.

DIMENSIONS OF SECOND MOMENTS OF MASS AND AREA. As we have already seen, the *second moment of mass* (or moment of inertia) of a solid body is equal to $\int mx^2$, and is therefore of the dimensions :—

$$[\mathrm{M}][\mathrm{L}]^2.$$

The *second moment of area* of an area is equal to $\int ax^2$, and is therefore of the dimensions :—

$$[\mathrm{L}]^4.$$

It is obvious from their dimensions that there is an essential difference between the two quantities.

CALCULATION OF SECOND MOMENTS OF AREA. The methods we employ for this purpose are very similar to those which we have explained for the determination of moments of inertia. We find the **area of an element** of the figure, choosing the element in such a way that we can express its area in terms of the geometry of the figure. Having found this expression for the element of area, we substitute it for the term a in the general expression $\int ax^2$, and integrate. Students who have worked through the earlier part of this chapter should find no difficulty in following the process, for it is simpler than the corresponding process for the determination of moments of inertia. It may be noted that we can obtain the *second moment of an area* from the *moment of inertia of a lamina* of the same size and shape by substituting A, the value of the area, for M the mass of the lamina : for example, in the case of a rectangular lamina, the moment of inertia about one edge is $\frac{1}{3}.\mathrm{M}.d^2$, where M is the mass of the lamina and d is its depth perpendicular to the axis. Substituting A for M, we

find that the second moment of a rectangular area about one edge is $\frac{1}{3}.A.d^2$, that is $\frac{1}{3}.(b.d)d^2$ or $\frac{1}{3}.b.d^3$.

We can obtain the same result from first principles in the following way:—

Let d = depth of area (see Fig. 123, page 282),
and b = breadth of area.

Area of elementary strip parallel to given axis

$$= b.dx.$$

Second moment of area of elementary strip about given axis

$$= b.x^2.dx.$$

Second moment of whole area about given axis

$$= b\int_0^d x^2.dx$$

$$= \tfrac{1}{3}.b.d^3 \qquad \text{as before.}$$

Similarly, for a circular area about a perpendicular axis through its centre, we have :—

Area of elementary ring (see Fig. 120, page 277)

$$= 2.\pi.x.dx.$$

Second moment of area of elementary ring about given axis

$$= 2.\pi.x^3.dx.$$

Second moment of whole area about given axis

$$= 2.\pi\int_0^r x^3.dx$$

$$= 2.\pi.\tfrac{1}{4}.r^4$$

$$= \tfrac{1}{2}.\pi.r^4$$

$$= \tfrac{1}{32}.\pi.d^4,$$

where d is the diameter of the circle.

It should be noticed that we usually express the moment of inertia of any body in terms of its *mass*, since the mass of a body can readily be determined; but the second moment of an area is more often expressed in terms of its *linear* dimensions *only*, such as its length, breadth, radius or diameter.

For example, the second moment of mass, or moment of inertia, of a circular disc about its own axis is $\frac{1}{2}.\mathrm{M}.r^2$ (page 277). The second moment of a circular area about a corresponding axis is $\frac{1}{2}.\mathrm{A}.r^2$, but it is more usual to state this in the form $\frac{1}{32}.\pi.d^4$, giving the result in terms of the diameter only.

POLAR SECOND MOMENTS. The moment of inertia of a lamina about an axis **perpendicular** to its plane is termed the **polar moment of inertia** of the lamina, and the second moment of an area about an axis **perpendicular** to its plane is termed the **second polar moment** of the area. The term *polar moment of inertia* may be extended also to solid bodies in such cases as that of a disc or cylinder about its own axis.

The Perpendicular Axes Theorem may then be stated in the form as follows:—*The polar moment of inertia of any lamina, about an axis passing through any point O in the lamina, is equal to the sum of the moments of inertia of the lamina about any two mutually perpendicular axes in the plane of the lamina and intersecting in the point O.*

SUMMARY OF CHAPTER 14

The **moment of inertia**, or **second moment of mass**, of a body about a given axis is equal to the sum of the products of the mass of each particle of the body into the square of its distance from that axis: that is:—

$$\mathrm{I} = \int mx^2.$$

It may also be defined as the product of the mass of the body and the square of its effective distance from (or radius of gyration about) that axis: that is:—

$$\mathrm{I} = \mathrm{M}k^2.$$

The **form** of the expression for the moment of inertia in any given case depends upon the **shape** of the body and the **position of the axis** with regard to it: that is, it depends upon the *distribution* of the material of the body with regard to the axis.

The **numerical value** of the moment of inertia in any given case depends also upon the **size** and **density** of the body, that is, it depends

upon the *nature* and *quantity* of the material composing the body, as well as its distribution.

To calculate the moment of inertia in any particular case, we find an expression for the mass of a suitable element of the body, substitute this expression for the term m in the general expression $\int mx^2$, and integrate.

Perpendicular Axes Theorem. The sum of the moments of inertia of any lamina about any two perpendicular axes in its plane, is equal to the moment of inertia of the lamina about an axis perpendicular to the lamina and passing through the point of intersection of the other two axes.

Parallel Axes Theorem. The moment of inertia of a body about any axis, is equal to the moment of inertia of the body about a parallel axis through its centre of mass, plus the product of the mass of the body and the square of the distance between the two axes: that is :—

$$I_x = I_G + Ml^2.$$

The **second moment of an area** about a given axis, is the sum of the products of each element of the area into the square of its distance from the given axis: that is :—

$$I = \int ax^2.$$

It may also be defined as the product of the area and the square of its effective distance from (or radius of gyration about) that axis: that is :—

$$I = Ak^2.$$

The *dimensions* of second moment of mass (or moment of inertia) are $[M][L]^2$. The dimensions of second moment of area are $[L]^4$.

The moment of inertia of a lamina, or the second moment of an area, about an axis perpendicular to its plane, is termed the **polar moment of inertia** or **second polar moment of area** respectively.

VALUES OF MOMENT OF INERTIA IN CERTAIN CASES

Solid Cylinder or Disc.	
About its own axis	$M.(\frac{1}{2}.r^2)$
About parallel axis at distance l . .	$M.(\frac{1}{2}.r^2 + l^2)$
About a diameter through centre of mass .	$M.(\frac{1}{4}.r^2 + \frac{1}{12}.l^2)$
About a diameter of one end . . .	$M.(\frac{1}{4}.r^2 + \frac{1}{3}.l^2)$
Hollow Cylinder or Disc.	
About its own axis	$M.(\frac{1}{2}r_1^2 + \frac{1}{2}r_2^2)$
Thin Rod.	
About perpendicular axis through centre .	$M.(\frac{1}{12}.l^2)$
About perpendicular axis through one end	$M.(\frac{1}{3}.l^2)$
Rectangular Lamina.	
About one edge	$M.(\frac{1}{3}.d^2)$
About parallel axis through centre . .	$M.(\frac{1}{12}.d^2)$
About perpendicular axis through centre .	$M.(\frac{1}{12}.d^2 + \frac{1}{12}.b^2)$
Square Lamina.	
About one edge	$M.(\frac{1}{3}.a^2)$
About parallel axis through centre . .	$M.(\frac{1}{12}.a^2)$
About perpendicular axis through centre .	$M.(\frac{1}{6}.a^2)$
About a diagonal	$M.(\frac{1}{12}.a^2)$
Triangular Lamina.	
About one edge	$M.(\frac{1}{6}.h^2)$
About parallel axis through centre . .	$M.(\frac{1}{18}.h^2)$
Parallelogrammic Lamina.	
About a diagonal	$M.(\frac{1}{6}.h^2)$
Circular Lamina.	
About its own axis	$M.(\frac{1}{2}.r^2)$
About a diameter	$M.(\frac{1}{4}.r^2)$
Circular Ring Lamina.	
About a diameter	$M.(\frac{1}{4}.r_1^2 + \frac{1}{4}.r_2^2)$
Rectangular Prism.	
About perpendicular axis through centre .	$M.(\frac{1}{12}.l^2 + \frac{1}{12}.b^2)$
Cube.	
About a diagonal of one face . . .	$M.(\frac{5}{12}.a^2)$
Thin Spherical Shell.	
About a diameter	$M.(\frac{2}{3}.r^2)$
Solid Sphere.	
About a diameter	$M.(\frac{2}{5}.r^2)$

EXAMPLES XIV

(*For Hints on Working Examples, see page* 21.)

1. *Find the moment of inertia of a solid disc fly-wheel, of* 85 *pounds mass and* 14 *inches in diameter, about its own axis.*

$$
\begin{aligned}
I &= \tfrac{1}{2}.M.r^2 \quad \text{(see page 277)} \\
&= \tfrac{1}{2}.85 \text{ pounds} \times (\tfrac{7}{12} \text{ foot})^2 \\
&= \frac{85 \times 49}{2 \times 144} \text{ pound-feet-square} \\
&= 14{\cdot}45 \text{ pound-feet-square.}
\end{aligned}
$$

2. *Determine the moment of inertia of a cast-iron ball,* 5 *inches in diameter, about a tangent. Density of cast-iron* = ·26 *pounds per cubic inch.*

$$
\begin{aligned}
\text{Mass of ball} &= \tfrac{4}{3}.\pi.r^3.\varrho \\
&= \tfrac{4}{3}.\tfrac{22}{7}.(2\tfrac{1}{2})^3.{\cdot}26 \\
&= \frac{55 \times 13}{42} \\
&= 17 \text{ pounds.}
\end{aligned}
$$

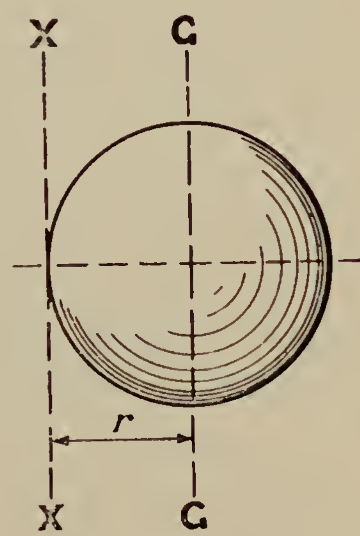


FIG. 140.

By the Parallel Axes Theorem :—

$$
\begin{aligned}
I_X &= I_G + M.r^2 \\
&= \tfrac{2}{5}.M.r^2 + M.r^2 \quad \text{(see page 294).} \\
&= \tfrac{7}{5}.M.r^2 \\
&= \tfrac{7}{5}.17.(\tfrac{5}{24})^2 \\
&= \frac{7 \times 17 \times 5}{24 \times 24} \\
&= 1{\cdot}03 \text{ pound-feet-square.}
\end{aligned}
$$

3. Find the moment of inertia, about its own axis, of a solid cast-iron fly-wheel, 2 feet 4 inches in diameter and 7 inches in thickness. Density of cast-iron = 450 pounds per cubic foot.

4. Determine the moment of inertia of a square lamina about an axis through one corner perpendicular to the plane of the lamina.

5. Find the moment of inertia of a cube about a diagonal through the middle points of two opposite edges.

6. Find the second moment of area of a triangle, of base b and perpendicular height h, about an axis through the apex parallel to the base.

7. Determine the second moment of area of an equilateral triangle, of side a, about the bisector of one angle.

8. Determine the second moment of area of an equilateral triangle of side a, about an axis in the plane of the triangle, parallel to the base, and half-way between the base and the apex.

9. Calculate the moment of inertia of a cube, of 4-inch edge and ·288 pound per cubic inch density, about a diagonal of one face. What would the moment of inertia be, about a parallel axis through the centre of gravity of the cube ?

10. Find the second polar moment of mass of an equilateral triangular lamina about an axis through its centre of gravity.

11. Find the moment of inertia of a cone, of height h and base radius r, about its own axis.

12. Find the second moment of area of a rectangle, 15 inches in depth and 4 inches in breadth, about an axis through the centroid of the rectangle parallel to the shorter side.

13. Find the second moment of area of the rectangle in the previous question, about an axis through the centroid parallel to the longer side.

14. A solid cylinder, of length l and radius r, has fixed to each end face, concentrically, a hemisphere of radius r. Find the moment of inertia of the whole body about the axis of the cylinder.

15. Find the moment of inertia of the body given in the previous question, about an axis through the centre of gravity and perpendicular to the axis of the cylinder.

16. What is the value, in gramme-centimetres-square, of the moment of inertia of a solid sphere of radius 5 centimetres and density 7·8 grammes per cubic centimetre about a diameter ?

17. The sides of a triangle are 4, 5, and 6 feet in length. Find the second moment of area of the triangle about each of the sides in turn.

18. Find the moment of inertia of a cube about one edge.

19. A square lamina, of edge a, has a circular hole, of radius r, cut from it, the centre of the hole being at the centre of the square. Find the moment of inertia of the remaining piece about a diagonal.

20. A rectangular prism is 10 inches in length, one inch in breadth, and half an inch in thickness ; find its moment of inertia about an axis parallel to the shortest edges and passing through the centre of gravity of the prism. Density of material = ·29 pound per cubic inch.

21. In the previous question, find the moment of inertia of the prism about an axis along one of the shortest edges.

22. Calculate the radii of gyration of the cube in question 9, about the two given axes.

23. What is the radius of gyration of the fly-wheel in question 3, about the given axis? What would it be about a diameter of one face?

24. Find the radii of gyration of the triangle given in question 17, about the given axes.

25. A cube, of edge a, has fixed to each face a hemisphere of radius $\frac{1}{2}a$. Find the moment of inertia of the whole body about an axis through the centroids of opposite faces of the cube.

26. The rim of a fly-wheel is 5 feet in outside diameter, 4 feet in inside diameter, and 8 inches in thickness, and is made of cast-steel of density 491 pounds per cubic foot. Neglecting the arms and boss, find the moment of inertia of the wheel about its own axis.

27. A cast-iron disc, 5 feet 6 inches in diameter and $7\frac{1}{2}$ inches in thickness, is fixed on a steel shaft 4 inches in diameter and 8 feet in length. Find the moment of inertia of the disc and shaft together, about their own axis. Density of cast-iron = ·26 pound per cubic inch: density of steel = ·288 pound per cubic inch.

28. A solid brass rod, one inch in diameter and 18 inches in length, is free to turn about an axis of symmetry perpendicular to its own axis. Find its moment of inertia about the axis of rotation. What would be the percentage error if it were treated as a thin rod? Density of brass = ·30 pound per cubic inch.

CHAPTER 15: PERIODIC MOTION

WHEN a body moves to and fro systematically, in such a way that every part of its motion recurs regularly, then it is said to move with *periodic motion*. There are many kinds of periodic motion, but by far the most frequently occurring and important is that which we term **Simple Harmonic Motion,** sometimes abbreviated to S.H.M.

A body, which moves backwards and forwards along a path in such a way that it always has an acceleration directed towards some fixed point in the path, and proportional to the distance of the body from the fixed point, measured along the path, is said to move with Simple Harmonic Motion.

For example, suppose that AOA′ is the path along which the body M moves to and fro (Fig. 141). If the motion is simple harmonic, then the *acceleration* of the body is always directed towards some fixed point O, and is proportional to the distance of M from O.

FIG. 141.

Now suppose that the body starts from rest at A, and moves towards O. As it moves along, its *velocity* increases, but its *acceleration,* being proportional to the distance of the body from O, decreases until it becomes zero at O. At this point the velocity has its maximum value, for when the body has passed O and is moving towards A′, the acceleration, being always towards O, is increasing in magnitude, but is in the opposite direction to that in which the body is moving and is therefore a retardation, so that the velocity is now decreasing.

Clearly the body will move towards A′ more and more slowly, until the acceleration has brought it to rest, and then, as it will still be under acceleration towards O, it will move back again towards O, with increasing

velocity, until, when it reaches O, it will be moving with maximum velocity towards A.

As soon as it has passed the point O, the acceleration, being always towards O, will be in the opposite direction to that of the velocity, and will therefore retard the motion of the body until at last it brings it to rest at the point A again, and then compels it to move back again towards O with velocity which again increases until O is reached, and then once more decreases as the body moves outwards towards A′.

Evidently the body, if not disturbed, will continue to move to and fro between the points A and A′, coming to rest momentarily at each of these points, and moving with maximum velocity as it passes the point O in either direction.

DERIVATION FROM CIRCULAR MOTION. Suppose that a body moves round a circular path with uniform speed u feet per second. Then if we look at the moving body with our line of sight in the plane of the circle, we shall see the body moving but the circle will not be visible. The body will appear to move, not in a circle, but to and fro along a straight line.

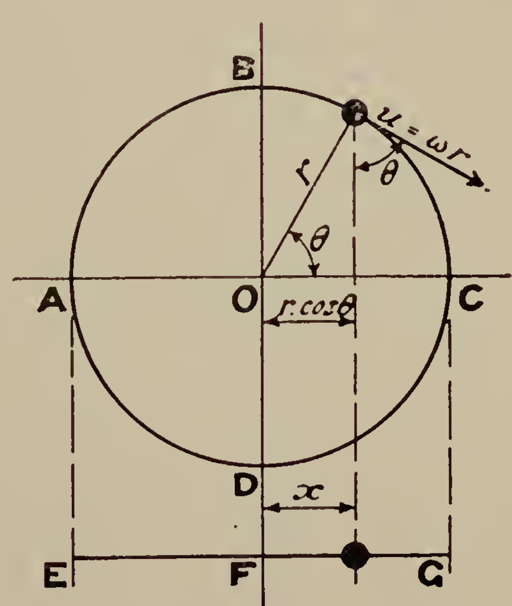


FIG. 142.

While the body moves round the circle from A to B (Fig. 142), it will appear to us to move along some straight line EF, where E is projected from A, and F is projected from B. Similarly, while the body moves round from B to C, it will appear to us to move from F to G. So, when the body travels round from C to D, we shall see it apparently moving from G back again to F, and while it moves from D to A, it will appear to move back from F to E.

Since the body is moving round the circle with

uniform speed, its angular velocity is also uniform, and equals say ω radians per second, so that the body must be subject to normal acceleration equal to $\omega^2.r$ feet per second per second, and to no other acceleration whatever.

We can resolve this acceleration parallel to AC and perpendicular to AC. The resolved part perpendicular to AC will have no effect upon the *apparent* motion along EFG. The apparent acceleration of the body will therefore be the resolved part along AC, that is, it will be :—

$$a = \omega^2.r.\cos\theta$$
$$= \omega^2.x,$$

where x is the apparent displacement of the body from its middle position F. The direction of this acceleration is, clearly, towards F (see Figs. 142 and 143).

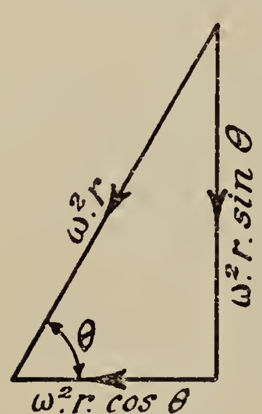


FIG. 143.

Now ω is a constant, so that ω^2 is also constant, and the acceleration is therefore directly proportional to the distance of the body from F, and is always directed towards F. Evidently the apparent motion of the body is simple harmonic. When a body moves along a straight line with S.H.M. we may therefore regard its motion as the projection of a uniform circular motion in a plane containing the straight line. This is frequently very convenient, but, of course, we must remember that in real cases of S.H.M. the body actually moves along the straight line, and it is the circular motion which is imaginary.

We ought here to notice that the definition of S.H.M. does not require that the path of the moving body shall be straight : it may be a curve of any kind so long as the acceleration of the body is directed tangentially towards some fixed point in it, and is proportional to the displacement from that fixed point, measured along the path. For the present, however, we shall confine our attention to simple harmonic motions which have straight paths.

We may verify the expression for the acceleration, $\omega^2.x$, by considering its dimensions. As the dimensions of any angular velocity ω are $[T]^{-1}$, and the dimensions of any distance x are $[L]$, therefore the dimensions of the acceleration $\omega^2.x$ are $[L][T]^{-2}$, which are the correct dimensions for a linear acceleration.

PERIODIC TIME. *The* **periodic time** *of a simple harmonic motion is the time occupied in moving from one end of the path to the other and back again.* In the case already considered, therefore, the periodic time is the time occupied in moving from E to G and back again to E. Now during this time the body will actually have moved completely round the circle, that is, through an angle of 2π radians, at an angular velocity of ω radians per second, so that the time occupied will be $2\pi/\omega$ seconds. Therefore the periodic time is :—

$$t = \frac{2\pi}{\omega} \text{ seconds.}$$

Now the acceleration, a, $= \omega^2.x$, so that $\omega^2 = \frac{a}{x}$, and $\omega = \sqrt{\frac{a}{x}}$. Therefore the periodic time

$$= t = \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{\sqrt{\frac{a}{x}}}$$

$$= 2\pi\sqrt{\frac{x}{a}}$$

$$= 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}} \text{ seconds.}$$

This is true for all simple harmonic motions and is therefore a very important result.

FORCE. Since the velocity of any body moving with simple harmonic motion is not constant but varies

according to the position of the body, so therefore the momentum of the body is also variable, and the body must (by the First Law of Momentum) be under the action of some external force.

If the mass of the body is constant, then the force acting upon it at any instant is (by the Second Law of Momentum) equal to the product of the mass of the body and the acceleration with which it is moving at that instant: that is, the force acting is :—

$$P = M.\omega^2.x \text{ poundals, (or dynes)}$$

where M is the mass of the body.

VELOCITY. The velocity of the body at the ends of the path is zero. The velocity at the middle point F of the path is the maximum velocity and is equal to the uniform speed u of the body moving round the circle, for at that point the motion of the body moving round the circle would be parallel with the straight line forming the path of the simple harmonic motion (see Fig. 142).

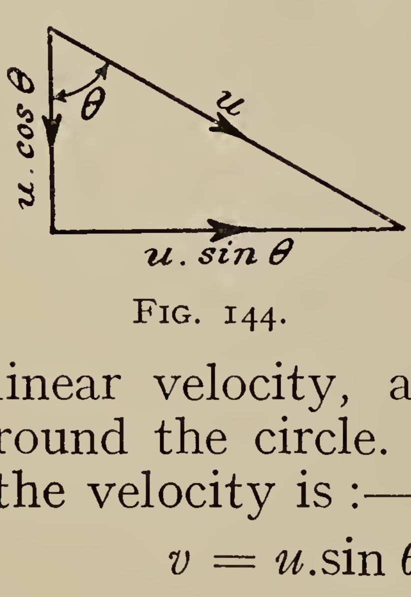


Fig. 144.

At any distance x from the centre of the path, the velocity is the resolved part along EFG of the linear velocity, at that instant, of the body moving round the circle. (See Figs. 142 and 144.) Therefore the velocity is :—

$$\begin{aligned} v &= u.\sin\theta \\ &= \omega.r.\frac{\sqrt{r^2 - x^2}}{r} \\ &= \omega.\sqrt{r^2 - x^2} \text{ feet per second per second.} \end{aligned}$$

AMPLITUDE. *The distance through which the body moves on either side of the fixed point* (that is, EF or FG) *is known as the* **amplitude** *of the motion.* It will be seen that the periodic time does not depend in any way upon the amplitude, for it varies only with variations of ω.

It should be realised that we can have any number

of different simple harmonic motions each with the same amplitude. We can see this at once by observing that if a body moves round a circular path it can have any one of an infinite number of different uniform speeds, and that corresponding to each of these we can obtain a simple harmonic motion, as shown above. The periodic times corresponding to the different speeds will, of course, vary inversely as ω.

STRENGTH OF CENTRE. From the definition of simple harmonic motion it is evident that, for any such motion :—

$$\frac{\text{acceleration}}{\text{displacement}} = \text{a constant.}$$

This constant ratio is known as the **strength of the centre** and is usually denoted by the Greek letter μ (mu). It will be seen that we have already obtained the value of this ratio as ω^2, where ω is the angular velocity of a body moving round a circle. Now if we have a real S.H.M. where there is no circular motion involved, ω is the angular velocity from which we may imagine the real linear velocity to be derived, and we shall probably not know its value. Therefore, instead of employing ω^2 to denote the strength of the centre, it is better in general cases to use μ.

We have, then, in any simple harmonic motion :—

Acceleration = displacement × strength of centre,

that is, $a = x.\mu$

so that Periodic time $= t$

$$= \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{\sqrt{\mu}} \text{ seconds.}$$

Since strength of centre is equal to acceleration divided by displacement, its **dimensions** will be the dimensions of a linear acceleration, **[L][T]**$^{-2}$, divided by the dimensions of a length, **[L]**, that is, they will be **[T]**$^{-2}$.

VIBRATION OF LOADED SPRINGS. If we hang a body, of say M pounds mass, on the lower end of a helical spring, as shown in Fig. 145, then the spring will extend under the force of gravity acting upon the body.

Fig. 145.

Suppose that the extension of the spring under the weight Mg poundals is l feet. It is known from experiment—this can easily be verified by the student—that the extension of such a spring is proportional to the load upon it, provided that the elastic limit of the material is not exceeded. Therefore if we now pull down the lower end of the spring a further x feet, then the elasticity of the spring will exert an upward force of M$g(x/l)$ poundals upon the suspended body, in addition to the force required to balance the weight of the latter.

As soon as the body is released, the unbalanced force upon it, M$g.\frac{x}{l}$ poundals, will cause it to move upwards with an acceleration of $g.\frac{x}{l}$ feet per second per second. Now g is a constant (at any given place), and l is also a constant, but x is the displacement of the body from its position of equilibrium, and this varies as the body moves upwards. Therefore the acceleration varies directly as x, which is the displacement, so that the motion is simple harmonic.

When the body reaches its position of equilibrium it will be moving upwards with considerable momentum: it will therefore pass this mean position and continue moving upwards. As it rises, however, its velocity will decrease, for the tension in the spring will lessen and will leave more and more of the weight, Mg, of the body unbalanced. Clearly this unbalanced weight will provide an increasing acceleration downwards towards the mean position.

When the body comes to rest at its highest position, it will immediately start moving down again, with increasing velocity and decreasing acceleration, and, passing its mean position, will gradually slow down until it again reaches its lowest position. These vibrations up and down would continue indefinitely but for the damping action of the air and other resistances.

It is interesting to consider the **changes in the energy** of the system during these vibrations. When the load is at its lowest point we may consider that the system has no *potential energy*. As the system is then momentarily at rest, it has no *kinetic energy* either. All the energy of the system is, in fact, stored in the spring as *strain energy*. As the load rises it acquires potential energy, and, as at the same time its velocity increases, it acquires kinetic energy also, the strain energy being correspondingly reduced. At the highest point the load is again at rest for the instant and therefore possesses no kinetic energy: its strain energy is also a minimum, so that now all the energy of vibration is potential.

The **time of a complete vibration** up and down is the periodic time of a simple harmonic motion, that is :—

$$t = 2\pi . \sqrt{\frac{\text{displacement}}{\text{acceleration}}}.$$

In this case the displacement at any instant is x feet, and the corresponding acceleration is, as we have seen, gx/l feet per second per second. Therefore :—

$$t = 2\pi . \sqrt{\frac{x}{gx/l}}$$

$$= 2\pi . \sqrt{\frac{l}{g}} \text{ seconds.}$$

The **strength of the centre** is :—

$$\mu = \frac{g}{l}.$$

The **maximum velocity** is :—

$$v = \text{amplitude} \times \sqrt{\mu}$$
$$= r.\sqrt{\frac{g}{l}} \text{ feet per second,}$$

where r (the amplitude of the motion) is the distance through which the body was pulled down before release.

SIMPLE PENDULUM. A simple pendulum is a very small but heavy body, attached to one end of a long weightless inextensible thread, and swinging to and fro in a vertical plane under the action of the force of gravity, the other end of the thread being attached to a fixed point.

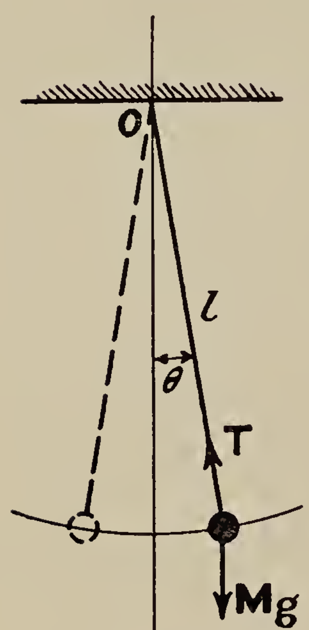


Fig. 146.

Suppose that M is the mass of the *bob* (as the suspended body is called), and l is the length of the thread. When the thread makes an angle θ with the vertical, as shown in Fig. 146, then the forces acting upon the body are its **weight,** Mg poundals vertically downwards, and the **tension,** T poundals along the string. We have therefore :—

Turning moment on body

$$= \text{force} \times \text{arm}$$
$$= \text{M}.g \times l.\sin\theta.$$

By the Second Law of Angular Momentum, this must be equal to the rate of change of angular momentum of the bob, that is, to I.a, where I is the moment of inertia of the bob about the axis of rotation, and a is its angular acceleration. Therefore :—

$$\text{I}.a = \text{M}.g.l.\sin\theta,$$

that is, $\text{M}.l^2.a = \text{M}.g.l.\sin\theta$ (since $\text{I} = \text{M}.l^2$ in this case),

so that $a = \frac{g}{l}.\sin\theta$ radians per second per second.

If the angle θ is very small, then $\sin\theta = \theta$ approximately, and we have :—

Angular acceleration of bob

$$=\frac{g}{l}.\theta,$$

so that the angular acceleration is directed towards the middle position of the pendulum and is proportional to the angular displacement, θ, from that position. The motion of a simple pendulum is therefore approximately simple harmonic: it is not, of course, strictly so, but the error involved is negligible, provided that the angle of swing remains small.

It should be noticed that, if a body, such as a pendulum, swings to and fro about a fixed axis, in such a way that the ratio

$$\frac{\text{angular acceleration}}{\text{angular displacement}}$$

is constant, then for any point in the body at a distance r from the axis of rotation,

$$\frac{\text{linear acceleration}}{\text{linear displacement}}=\frac{\text{angular acceleration}\times r}{\text{angular displacement}\times r}=\text{constant.}$$

Therefore every point of such a body moves with simple harmonic motion.

The **periodic time** of a simple pendulum is given by:—

$$t=2\pi.\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

$$=2\pi.\sqrt{\frac{\theta}{g\theta/l}}$$

$$=2\pi.\sqrt{\frac{l}{g}}\text{ seconds.}$$

The time of oscillation is quite independent, therefore, of the mass of the bob and of the amplitude of the swing (provided that the angle of swing is small) and depends only on the length of the thread, and the magnitude of the acceleration due to gravity.

Every student should experiment for himself with a simple pendulum, and use it to determine as accurately

as possible the magnitude of the acceleration due to gravity. The length of the pendulum should first be measured accurately, and then a determination should be made of the time required for, say, 100 complete oscillations: from this can be calculated the periodic time t of the pendulum, and hence the value of g found. It should be noted that in addition to the very small error involved in assuming that the motion is simple harmonic, there are also slight errors caused by assuming that the thread is weightless, and by the difficulty of measuring the effective length of the thread correctly.

COMPOUND PENDULUM. If a body of *any* shape is suspended at a fixed axis and allowed to swing to and fro under the action of gravity, it behaves as a pendulum; that is, any point in it moves with simple harmonic motion, provided that the angle of swing is small. Such an oscillating body is termed a **compound pendulum.**

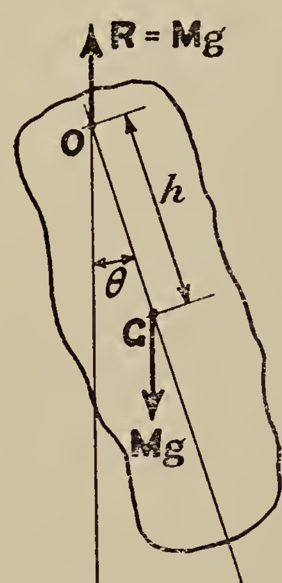


FIG. 147.

The point at which the axis of suspension cuts the vertical plane in which the centre of gravity of the body swings to and fro, is termed the **centre of suspension.**

Let O (Fig. 147) represent the centre of suspension and G the centre of gravity of the body, and let the distance OG be h. Then when the line OG is at an angle θ with the vertical, the turning moment on the body, tending to restore it to its position of equilibrium, is:—

$$T = M.g \times h.\sin\theta$$
$$= M.g.h.\theta \text{ approximately, if } \theta \text{ is small.}$$

By the Second Law of Angular Momentum, this must be equal to the rate of change of angular momentum of the body, that is, it must be equal to $I.\alpha$, where I is the moment of inertia of the body about the axis of suspension, and α is the angular acceleration of the body.

Therefore :—

$$\mathrm{I}.\alpha = \mathrm{M}.g.h.\theta,$$

whence $$\alpha = \frac{\mathrm{M}.g.h.\theta}{\mathrm{I}}$$

$$= \frac{\mathrm{M}.g.h.\theta}{\mathrm{M}.k^2}$$ (where k = radius of gyration of body about axis of suspension)

$$= \frac{g.h}{k^2}.\theta \text{ radians per second per second,}$$

so that the angular acceleration, α, is directed towards the middle position of the pendulum and is proportional to the angular displacement, θ, from that position. The motion of a compound pendulum is therefore approximately simple harmonic, provided that the angle of swing is small.

The **periodic time** of a compound pendulum is given by :—

$$t = 2\pi.\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

$$= 2\pi.\sqrt{\frac{\theta}{g.h.\theta/k^2}}$$

$$= 2\pi.\sqrt{\frac{k^2}{g.h}} \text{ seconds.}$$

SIMPLE EQUIVALENT PENDULUM. We have now seen that the periodic time of a *simple* pendulum is:—

$$= 2\pi.\sqrt{\frac{l}{g}} \text{ seconds,}$$

where l is the length of the pendulum ; and that the periodic time of a *compound* pendulum is :—

$$= 2\pi.\sqrt{\frac{k^2}{g.h}} \text{ seconds,}$$

where k is the radius of gyration of the pendulum about its axis of suspension, and h is the distance from the centre of suspension to the centre of gravity of the pendulum. There is therefore a close resemblance between the two expressions, the difference between them

being that l in the simple pendulum is replaced by k^2/h in the compound pendulum.

Clearly, if we had a simple pendulum such that its length l was equal to the quantity k^2/h for a given compound pendulum, then the periodic times of the two pendulums would be equal, and the simple pendulum would be what we call the **simple equivalent pendulum** of the given compound pendulum.

We may say, then, that for any compound pendulum, the *simple equivalent pendulum* is a simple pendulum whose periodic time is the same as that of the compound pendulum, and whose length is equal to the square of the radius of gyration of the compound pendulum about its axis of suspension, divided by the distance from that axis of the centre of gravity of the compound pendulum.

If k_G is the radius of gyration of the compound pendulum about an axis through the centre of gravity parallel to the axis of suspension, then, by the Parallel Axes Theorem (page 286), we have :—

$$M.k^2 = M.k_G{}^2 + M.h^2,$$

whence $$k^2 = k_G{}^2 + h^2,$$

so that the periodic time of a compound pendulum may be stated as :—

$$t = 2\pi.\sqrt{\frac{k_G{}^2 + h^2}{g.h}} \text{ seconds.}$$

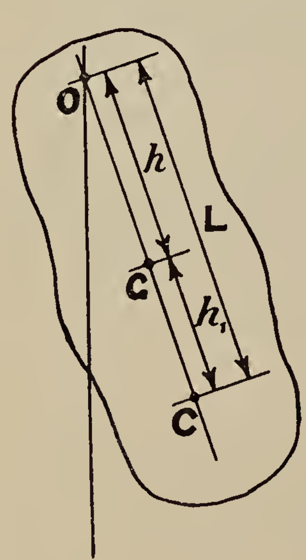


Fig. 148.

CENTRES OF SUSPENSION AND OSCILLATION. We have already seen that the point at which the axis of suspension cuts the vertical plane in which the centre of gravity of the compound pendulum swings to and fro, is termed the **centre of suspension.**

The point C in the line OG produced, such that OC is equal to the length of the simple equivalent pendulum, is termed the **centre of oscillation.**

Let OC = L = length of S.E.P.

Then we have :—

$$L = \frac{k_G^2 + h^2}{h}$$

$$= \frac{k_G^2}{h} + h,$$

so that

$$GC = \frac{k_G^2}{h}$$

$$= h_1 \text{ say, so that } k_G^2 = h.h_1.$$

Now suppose that the pendulum is reversed, and allowed to swing about an axis through C parallel to the original axis through O. Then the length of the simple equivalent pendulum when the body is swinging about this new axis

$$= L_1 = \frac{k_G^2 + h_1^2}{h_1}$$

$$= \frac{k_G^2}{h_1} + h_1$$

$$= \frac{h.h_1}{h_1} + h_1$$

$$= h + h_1 = L,$$

so that the simple equivalent pendulum and the periodic time are the same when the pendulum swings about the axis through C as when it swings about the axis through O ; that is, *the centres of suspension and oscillation are interchangeable.*

It should be noticed that, as $k_G^2 = h.h_1$, the lengths h and h_1 are only equal when each is equal to k_G. It is only in this exceptional case that the centre of oscillation and the centre of suspension will be equidistant from the centre of gravity of the pendulum.

KATER'S PENDULUM. If we find the periodic time of a compound pendulum about any axis, and then determine the position of a parallel axis, on the other side of the centre of gravity, such that the periodic time about the new axis is equal to that about the original axis,

then, *provided that the two axes are not equidistant from the centre of gravity*, if either axis be taken to pass through the centre of suspension, the other will pass through the centre of oscillation and vice versa ; that is, the distance between the two axes is equal to the length of the simple equivalent pendulum.

This is the principle of **Kater's Pendulum,** invented by Captain Henry Kater, an English army officer and scientist, in the early part of the nineteenth century.

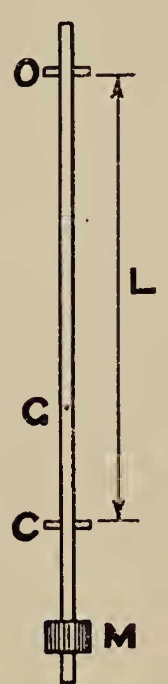


Fig. 149.

One form of this pendulum consists of a bar of metal containing two knife-edges, as shown at O and C in Fig. 149. These knife-edges are not equidistant from the centre of the bar. The bar is also provided with an adjustable mass M, which can be fixed in any required position on the bar.

By altering the position of M, the periodic times about O and about C are both altered, and when M is in such a position that these are equal, then the distance L, between the axes, is equal to the length of the simple equivalent pendulum. As we can measure this length very accurately, we have in Kater's pendulum a much more precise means of determining the value of g than is given by the simple pendulum.

Clearly the periodic time is :—

$$t = 2\pi . \sqrt{\frac{L}{g}} \text{ seconds (where L = length of S.E.P.),}$$

whence $t^2 = 4\pi^2\dfrac{L}{g}$

and $$g = \frac{4.\pi^2.L}{t^2} \text{ feet per second per second.}$$

MOTION OF STEAM-ENGINE PISTON. If the crank of an ordinary reciprocating steam-engine rotates with constant speed, then the motion of the cross-head, piston-rod, and piston is approximately simple harmonic.

If the connecting-rod were infinitely long, then the motion of the reciprocating parts would be truly simple harmonic, the connecting-rod being always in the same straight line as the piston-rod.

In the real engine the connecting-rod is comparatively short, and its obliquity to the line of stroke is by no means negligible. The result of this obliquity is that the acceleration of the piston is not exactly proportional to the distance from the middle point of its path, being slightly greater at the beginning of the forward stroke than it would be for simple harmonic motion, and slightly less at the end of the stroke. Also the acceleration becomes zero at a point which is not the middle point of the path, although near to it.

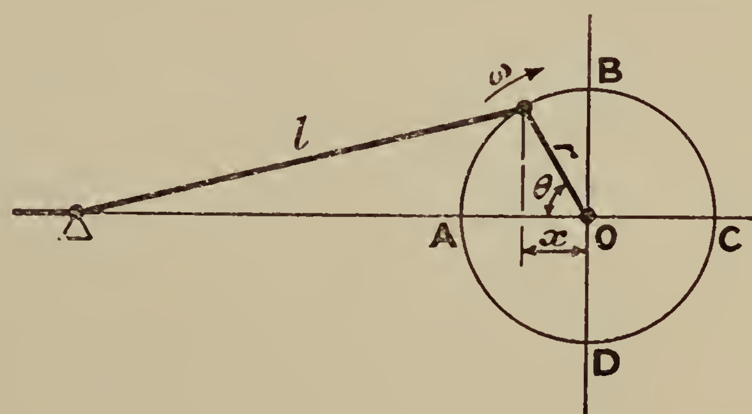


FIG. 150.

Suppose that r (Fig. 150) is the radius of the crank, and l the length of the connecting-rod, and that the angular velocity of the crank is ω radians per second. Then if the motion were simple harmonic, the acceleration of the reciprocating parts of the engine would be :—

$$a = \omega^2.r.\cos\theta$$
$$= \omega^2.x \quad \text{as before.}$$

This is the resolved part of the acceleration of the crank-pin, in the direction of motion of the piston and cross-head, *accurately*, but, owing to the obliquity of the connecting-rod, it is not transmitted exactly to those parts.

SUMMARY OF CHAPTER 15

A body which moves backwards and forwards along a path in such a way that it always has an acceleration directed towards some fixed point in the path, and proportional to the distance of the body from the fixed point, measured along the path, is said to move with **Simple Harmonic Motion.**

Simple harmonic motion may be regarded as the projection of a uniform circular motion on to a straight line in the plane of the circle.

The **periodic time** of a simple harmonic motion is the time occupied in moving from one end of the path to the other and back again. The periodic time is given by:—

$$t = 2\pi . \sqrt{\frac{\text{displacement}}{\text{acceleration}}}.$$

The constant ratio of *acceleration* to *displacement* for any given S.H.M., is known as the **strength of the centre,** and is denoted by μ, so that the acceleration $= a = \mu . x$, where x is the displacement at any instant.

The **force** acting on the body at any instant $= M.\mu.x$.

The distance through which the body moves on either side of the fixed point is known as the **amplitude** of the motion.

The **velocity** of the body at any point of its path is equal to $\sqrt{\mu . (r^2 - x^2)}$, where r is the amplitude of the motion.

If a heavy body is hung on the lower end of a helical spring, and allowed to vibrate up and down, it moves with simple harmonic motion.

A **simple pendulum** is a very small but heavy body, attached to one end of a long weightless inextensible thread, and swinging to and fro in a vertical plane, the other end of the thread being attached to a fixed point. The motion of such a body is approximately simple harmonic, provided that the angle of swing is small.

A **compound pendulum** is a body of any shape which is allowed to swing to and fro, about a horizontal axis, under the action of gravity. If the angle of swing is small, then the motion is approximately simple harmonic.

The simple pendulum which would have the same periodic time as a given compound pendulum, is termed the **simple equivalent pendulum** for that compound pendulum. The length L of the simple equivalent pendulum, is equal to k^2/h, where k is the radius of gyration of the compound pendulum about its axis of suspension, and h is the distance of that axis from the centre of gravity of the pendulum.

The point at which the axis of suspension of a compound pendulum cuts the vertical plane in which the centre of gravity swings to and fro, is termed the **centre of suspension.** The point in the line through the centre of suspension and the centre of gravity, such that its distance from the centre of suspension is equal to the length of the simple equivalent pendulum, is termed the **centre of oscillation.** The centres of suspension and oscillation are interchangeable.

Kater's Pendulum is a special form of compound pendulum so arranged that the length of the corresponding simple equivalent pendulum can readily be determined. It gives us a means of determining the value of the acceleration due to gravity with great accuracy.

The motion of the reciprocating parts of a steam-engine is approximately simple harmonic. If the speed of the crank-pin were uniform,

and the connecting-rod infinitely long, then the motion would be accurately simple harmonic. In any real engine the obliquity of the connecting-rod causes the motion to deviate more or less from simple harmonic.

EXAMPLES XV

(*For Hints on Working Examples, see page* 21.)

1. *The periodic time of a simple harmonic motion is* 2·9 *seconds, and the maximum acceleration is* 8 *feet per second per second. Find the amplitude of the motion, the maximum velocity, and the strength of the centre.*

Let t = periodic time, in seconds.
and μ = strength of centre, in F.P.S. units.

Then $$t = \frac{2\pi}{\sqrt{\mu}},$$

so that $$\mu = \left(\frac{2\pi}{t}\right)^2$$
$$= \left(\frac{44}{7 \times 2{\cdot}9}\right)^2$$
$$= 4{\cdot}7 \text{ F.P.S. units.}$$

Amplitude of motion
$$= \frac{\text{maximum acceleration}}{\text{strength of centre}}$$
$$= \frac{8 \text{ feet per second per second}}{4{\cdot}7 \text{ F.P.S. units}}$$
$$= 1{\cdot}7 \text{ feet.}$$

Maximum velocity
$$= \text{amplitude} \times \sqrt{\mu}$$
$$= 1{\cdot}7 \text{ feet} \times \sqrt{4{\cdot}7 \text{ units}}$$
$$= 1{\cdot}7 \times 2{\cdot}17$$
$$= 3{\cdot}69 \text{ feet per second.}$$

2. *The strength of the centre of a simple harmonic motion is* 12 *F.P.S. units, and the velocity at the middle point of the path is* 2 *feet per second. Find the periodic time, and the amplitude of the motion.*

Periodic time $$= \frac{2\pi}{\sqrt{\text{strength of centre}}} \text{ seconds}$$
$$= \frac{2\pi}{\sqrt{12}} \text{ seconds}$$
$$= 1{\cdot}814 \text{ seconds.}$$

Maximum velocity (i.e., velocity at middle point of path)

$$= \text{amplitude} \times \sqrt{\text{strength of centre}},$$

whence

$$\text{amplitude} = \frac{\text{maximum velocity}}{\sqrt{\text{strength of centre}}} \text{ feet}$$

$$= \frac{2 \text{ feet per second}}{\sqrt{12}}$$

$$= \cdot 577 \text{ foot}$$

$$= 6{\cdot}92 \text{ inches.}$$

3. *A helical spring, hanging vertically, is extended* $3\frac{1}{4}$ *inches when a load of* 4 *pounds-weight is suspended from its lower end. If the load is pulled down a further* 2 *inches and then released, find the periodic time in which it makes a complete vibration, and the velocity with which it passes its middle position. Neglect the mass of the spring.*

$$\text{Periodic time} = 2\pi.\sqrt{\frac{\text{static extension}}{g}}$$

$$= 2\pi.\sqrt{\frac{3\frac{1}{4}}{12 \times 32{\cdot}2}}$$

$$= 2\pi{\cdot}\sqrt{\cdot 00842}$$

$$= \cdot 576 \text{ second.}$$

Maximum velocity (i.e., velocity of passing middle position)

$$= \text{amplitude} \times \sqrt{\text{strength of centre}}$$

$$= \text{amplitude} \times \sqrt{\frac{g}{\text{static extension}}}$$

$$= \tfrac{1}{6} \text{ foot} \times \sqrt{32{\cdot}2 \times 48/13}$$

$$= \tfrac{1}{6} \times \sqrt{119}$$

$$= \tfrac{1}{6} \times 10{\cdot}9$$

$$= 1{\cdot}82 \text{ feet per second.}$$

4. *A simple pendulum,* 217 *centimetres in length, makes* 100 *complete oscillations in* 296 *seconds. Determine the value of g in both C.G.S. and F.P.S. units.*

Periodic time of simple pendulum

$$= 2\pi\sqrt{\frac{\text{length}}{g}}.$$

In this case the periodic time is $\frac{296}{100} = 2{\cdot}96$ seconds: therefore :—

$$2{\cdot}96 = 2\pi\sqrt{\frac{217}{g}}.$$

Squaring both sides, we have :—

$$8{\cdot}76 = 4\pi^2 \times \frac{217}{g},$$

whence
$$g = \frac{4 \times 9{\cdot}87 \times 217}{8{\cdot}76}$$
$$= 979 \text{ centimetres per second per second.}$$

Also, since one centimetre equals ·0328 foot, we have :—

$$g = (979 \times {\cdot}0328) \text{ feet per second per second}$$
$$= 32{\cdot}1 \text{ feet per second per second.}$$

5. *A bar of brass,* 1 *inch square and* 5 *feet in length, is suspended at a point* 1 *foot from one end and swings to and fro under the action of gravity. Find its periodic time, and the distance of the centre of oscillation from the other end.*

Let k_G = radius of gyration of the bar about a perpendicular axis through the centre of gravity,

and h = distance between centre of suspension and centre of gravity.

Then the length of the simple equivalent pendulum

$$= \text{L} = \frac{k_G{}^2 + h^2}{h}.$$

Let l = length of bar = 5 feet,

and b = breadth of bar = $\frac{1}{12}$ foot.

Then we have :—

$$k_G{}^2 = \tfrac{1}{12}(l^2 + b^2)$$
$$= \tfrac{1}{12}(25 + {\cdot}007)$$
$$= 2{\cdot}084 \text{ feet}^2,$$

so that
$$\text{L} = \frac{2{\cdot}084 + 2{\cdot}25}{1{\cdot}5} \quad (\text{since } h = 1{\cdot}5 \text{ feet})$$
$$= 2{\cdot}89 \text{ feet.}$$

$\therefore$ Periodic time
$$= 2\pi.\sqrt{\frac{\text{L}}{g}}$$
$$= 2\pi.\sqrt{\frac{2{\cdot}89}{32{\cdot}2}}$$
$$= 1{\cdot}884 \text{ seconds.}$$

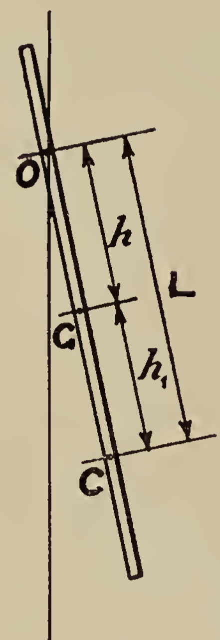


Fig. 151.

Distance of centre of oscillation from lower end of bar

$$= \tfrac{1}{2}l + h - \text{L (see Fig. 151)}$$
$$= 2{\cdot}5 + 1{\cdot}5 - 2{\cdot}89$$
$$= 1{\cdot}11 \text{ feet}$$
$$= 13{\cdot}32 \text{ inches.}$$

6. The periodic time of a simple harmonic motion is $3\frac{1}{2}$ seconds, and the maximum velocity is 5 feet per second. Find the amplitude of the motion, the strength of the centre, and the maximum acceleration.

7. The strength of the centre of a simple harmonic motion is 33 F.P.S. units, and the amplitude is 8 inches. Find the periodic time and the velocity at the middle of the path.

8. The acceleration at the end of the path of a simple harmonic motion is 15 feet per second per second, and the velocity is 3 feet per second when the displacement is $\sqrt{22}$ feet. Find the amplitude and periodic time.

9. The periodic time of a simple harmonic motion is $1\frac{3}{4}$ seconds and the acceleration is 3 feet per second per second half-way between the middle and end of the path. Find the amplitude.

10. A loaded helical spring has a statical extension of $4\frac{1}{2}$ inches. If it is made to vibrate up and down, what will be its periodic time? If the amplitude of the vibrations is 3 inches, what is the velocity when the displacement is 2 inches?

11. A helical spring extends 5 inches when a load of 7 pounds-weight is hung from its lower end. If a 4-pound weight is substituted for the 7-pound weight, and the loaded spring is made to vibrate vertically, what will be its periodic time?

12. Find the number of vibrations per minute that will be made by a helical spring carrying a load of 6 pounds-weight, if the same spring when loaded with $3\frac{1}{4}$ pounds-weight makes 91·6 vibrations per minute.

13. If g is equal to 981 centimetres per second per second, what will be the length of the simple pendulum which will make 24 complete oscillations per minute?

14. Calculate the value of g, given that a simple pendulum 5 feet 6 inches in length makes 100 complete oscillations in $260\frac{1}{2}$ seconds.

15. The moment of inertia of a compound pendulum about its axis of suspension is 4·37 pound-feet-square and its mass is 4 pounds.

If the distance from the centre of gravity of the pendulum to the centre of suspension is 9 inches, what is the periodic time ? What length would a simple pendulum have to be made in order to have the same time of oscillation ?

16. A steel disc, 15 inches in diameter, is pivoted at a point $6\frac{1}{4}$ inches from its centre and allowed to swing to and fro in its own plane under the action of gravity. What is its periodic time ?

17. A wooden ball, $2\frac{1}{2}$ inches in diameter, is used as the bob of an approximate simple pendulum. If the distance from the centre of the ball to the centre of suspension is ten inches, what will be the percentage error involved in calculating its periodic time as that of a simple instead of a compound pendulum ?

18. A cylindrical rod, 2 feet long and 2 inches in diameter, is free to swing about a horizontal axis at right angles to its geometrical axis and intersecting it. Find the position of the axis of suspension if the length of the equivalent simple pendulum is a minimum. (*Univ. Lond. Int. Eng.* 1920.)

19. How many complete oscillations per minute will be made by a cube of cast-iron weighing 20 pounds, if it is allowed to swing freely, under the action of gravity, about a horizontal axis along one edge ? Density of cast-iron = 450 pounds per cubic foot.

20. Find the energy of vibration of each of the loads on the spring in question 12, if the amplitude is 3 inches in each case.

21. If the vibrating body in question 6 has a mass of 10 pounds, what is its energy of vibration ?

22. The periodic time of a simple harmonic motion is 5 seconds, the mass of the oscillating body is 14 pounds, and its vibration energy is 120 foot-pounds. Find the amplitude of the motion, and the acceleration half-way from the middle of the path to the end.

CHAPTER 16: GYROSCOPIC MOTION

WE have now to deal with some curious phenomena which we encounter when studying the motion of rotating bodies. Some of these phenomena are so familiar that we accept them unquestioned as *facts*, although at first sight they seem to contradict the principles of Mechanics with which we have already become acquainted. Let us consider some instances.

A boy amuses himself by spinning a **peg-top.** By means of a string wound round the top and then quickly unwound, he sets the top rotating rapidly round its own axis of symmetry. The top, which, if it were not spinning, could not be made to stand upright and unsupported on its peg, when it *is* spinning will remain in this position, and seems indeed to resist any effort to overturn it (Fig. 152).

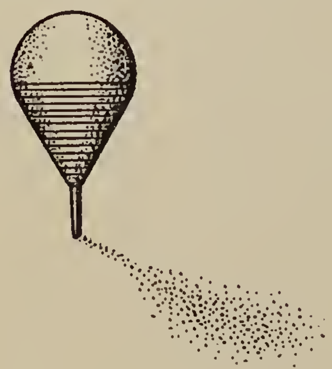


FIG. 152.

Why does the spinning top behave in this curious way? Why does not the force of gravity cause it to fall to the ground, instead of standing on end? To say that it is because it is spinning is no answer. What we have to determine is *why* the fact that the top is spinning should make it apparently immune from the ordinary laws of gravity.

Again, a child bowls its **hoop** along the road. So long as the hoop is rolling fairly fast it does not fall to the ground, even if left to itself and even if it leans over to one side or the other. In the latter case it rolls towards the side to which it is leaning, but it does not fall (Fig. 153). Why should the hoop appear to defy the attraction

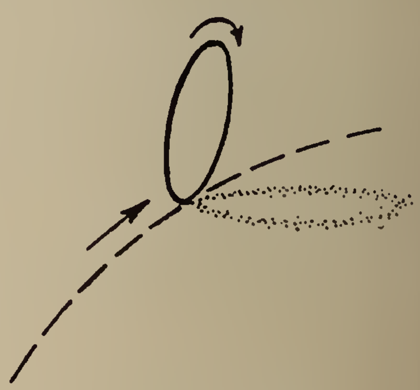


FIG. 153.

of the earth, which acts on all material bodies? If the same hoop were to be left to itself when it was not rolling, we know that it would at once fall to the ground.

Let us consider another experiment: if possible, the student should perform it for himself. If the **wheel of a bicycle** is removed from the frame, held in a vertical plane, and caused to rotate rapidly about its own axis, it will be found that any attempt to turn the wheel out of the plane in which it is spinning will be resisted, and the wheel will give a curious wriggle which may even wrench it out of our hands. Yet to turn the same wheel over in the same way when it is not rotating is perfectly simple and easy.

Another instructive experiment with a bicycle wheel was devised by Sir George Greenhill. The wheel—the largest and heaviest available—is mounted on a special axle which projects six or eight inches on each side. Collars on the axle hold the wheel in place but leave it free to turn easily. One end of the axle is firmly fixed in a hard wood ball about 2½ inches in diameter, as shown in Fig. 154.

As a preliminary, the student should try to keep the wheel upright by holding the ball in one hand only: it will be found difficult, if not impossible, to do this.

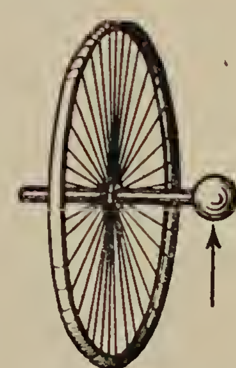


FIG. 154.

To start the experiment the student should hold the ball lightly in one hand, say the left, the other end of the axle being supported by someone else. With his free hand he should set the wheel spinning, giving it as high a speed as possible.

It will now be found that, if the other end of the axle is released, the student can easily support the wheel, and it will remain practically upright, the ball merely resting in his hand, and the other end of the axle being quite unsupported.

This seems a contradiction of the laws of gravity even more flagrant than the other cases to which we have referred. The bicycle wheel, spinning merrily round and maintaining its upright position, although one

end of its axle is unsupported, appears quite unaffected by any tendency to fall.

If, however, the student actually performs the experiment, instead of merely reading about it, he will quickly notice another curious fact. He will find that the wheel, in addition to its spin, will tend to rotate slowly *about him,* so that he is obliged to turn slowly about his own vertical axis. If he turns at the right speed he will find that the ball will rest in his hand without requiring any grip. As the wheel slows down he will find that he has to turn faster and faster, until at length he has to give up and let the wheel fall to the ground or apply his other hand to it. In this second rotation, which is termed **Precession,** we have the clue to the curious behaviour of the wheel.

All these phenomena are instances of what we term **Gyroscopic Motion.** But, before we investigate them further, there are certain laws and rules about which we must be quite sure, viz.,

VECTOR QUANTITIES OF THE SECOND CLASS. See Chapter 7, page 121, and Chapter 13, page 258.

THE LAWS OF ANGULAR MOMENTUM. See Chapter 13, page 256.

If the student is not perfectly clear about these very important matters, he should refer back to the appropriate chapters and refresh his memory.

Gyroscopic phenomena may be defined as *those which occur when it is attempted to turn the axis about which a body is spinning, into a new position at an angle with its original position.*

Instances of gyroscopic motion are of very frequent occurrence : we have noticed a few above, but there are very many more to be found in everyday life. In this book we cannot treat the subject exhaustively, so we will take one case for the purpose of explanation and discussion, and leave the student to apply similar reasoning to other instances he can find. He should notice

carefully that any action which involves *turning the axis of spin of a body through an angle* into a new position, must involve phenomena of this type.

THE GYROSCOPE. The particular case which we will take is that of a **fly-wheel,** supported so that it is free to turn in any direction, as explained below. For convenience, we will term the geometrical axis of the fly-wheel its **axle,** and its rotation *about this particular axis* its **spin.** The student should notice very carefully the particular way in which we here employ these terms. Also for convenience we shall usually take the axle as being horizontal, but it must be remembered that similar results might be obtained with the axle in any other direction.

Suppose, then, that the fly-wheel is spinning with uniform angular velocity, ω radians per second, its axle being horizontal, as shown by AA in Fig. 155. Then if I is the moment of inertia of the fly-wheel about its axle, the angular momentum of the wheel is $I\omega$.

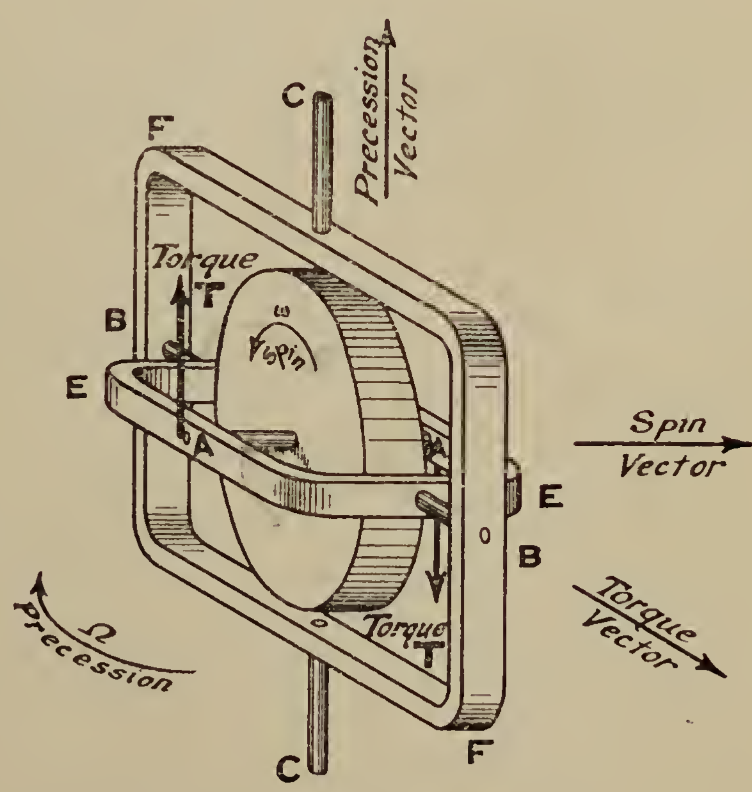


FIG. 155.

Suppose, also, that the axle AA is supported by the frame EE as shown, and that this frame is itself carried in such a way that it can turn freely about an axis BB which is perpendicular to AA, and in the same horizontal plane, the axis BB being itself supported by the frame FF, which is free to turn about the vertical axis CC. It will be seen that three separate rotations of the wheel are possible, the three **axes of rotation,** AA,

BB, and CC, being (initially) mutually perpendicular. The frames EE and FF are termed **gimbals,** and the whole apparatus a **gyroscope.**

Now suppose that, while the wheel is spinning, a torque T is acting on the ends of the axle AA, tending to cause the wheel and axle, together with the gimbal EE, to rotate about the other horizontal axis BB. One instinctively expects that the wheel and gimbal will actually rotate about BB, but if we try the experiment we find that, instead, the whole apparatus rotates about the *vertical* axis CC, the wheel continuing meanwhile its original spin about its axle AA.

This apparently anomalous result is quickly explained when we draw the **Parallelogram of Angular Momenta,** as shown in Fig. 156.

aa_1 represents the original angular momentum Iω, about the axle AA. aa_2 represents the angular momentum about the axis BB, which will be generated by the torque T in any very short interval of time *dt*. This added momentum will be equal to T.*dt*, since change of angular momentum = torque producing change $\times$ time during which it acts (see page 261).

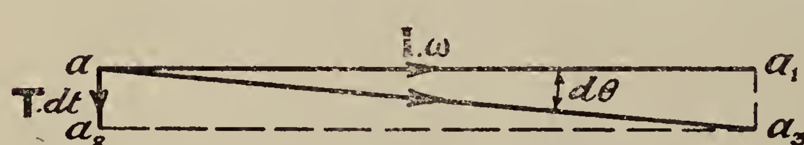


Fig. 156.

The **resultant** of these two angular momenta, Iω and T.*dt*, is found by completing the parallelogram and drawing the diagonal aa_3. This diagonal aa_3 then represents the angular momentum of the fly-wheel after the short interval of time *dt*. Now aa_1 and aa_2 are both in the same horizontal plane: therefore their resultant aa_3 is also in that horizontal plane, so that after the interval *dt* the wheel will still be spinning with its axle horizontal, but the axle will be at an angle $d\theta$ with its original position. In other words, what will actually happen is that, instead of the wheel rotating about the axis BB under the action of the torque T, it rotates about the perpendicular axis CC; which is exactly the result obtained experimentally.

PRECESSION. This rotation about the *vertical* axis CC, under the influence of a couple acting about the *horizontal* axis BB, is, as already noted in the case of the bicycle wheel, given the name of **Precession,** and the fly-wheel is said to **precess** about the axis CC under the action of the torque T.

In each indefinitely small interval of time dt the wheel turns through an indefinitely small horizontal angle $d\theta$ about the vertical axis CC. Therefore the rate at which it turns, that is, the **angular velocity of precession,** is :—

$$\Omega = \frac{d\theta}{dt} \text{ radians per second.}$$

The symbol Ω is the capital form of the Greek letter omega.

GYROSCOPIC COUPLE. The change of angular momentum in the time dt is $\mathrm{T}.dt$, and this is represented by aa_2 which is clearly equal to a_1a_3. But, since $d\theta$ is an indefinitely small angle :—

$$a_1a_3 = aa_1.d\theta,$$

so that $$\mathrm{T}.dt = \mathrm{I}.\omega.d\theta$$

and $$\mathrm{T} = \mathrm{I}.\omega.\frac{d\theta}{dt}$$

$$= \mathrm{I}.\omega.\Omega.$$

This is a very important result. Putting it into words, we may say that *the torque required to maintain an angular velocity of precession Ω in a body which is spinning about its axle with angular momentum* $\mathrm{I}.\omega$ *is equal to* $\mathrm{I}.\omega.\Omega$.

The torque T causing the precession is generally termed the **Gyroscopic Couple** or the **Precessional Torque.**

We may therefore put the results we have obtained in the following form :—

Gyroscopic Couple
= Moment of Inertia of body about its axle × Angular Velocity of Spin × Angular Velocity of Precession.

$$\textbf{Angular Velocity of Precession} = \frac{\text{Gyroscopic Couple}}{\text{Angular Momentum of body about its axle}}.$$

The case of the bicycle wheel with its axle supported at one end only will now be readily understood. Here the gyroscopic couple is provided by the weight of the wheel acting downwards and the reaction of the hand acting upwards. This couple tends to turn the wheel about a horizontal axis at right angles to the axle, and if the wheel is not spinning about the axle, it actually does so turn it. But when the wheel *is* spinning about its axle, then the angular momentum generated by the gyroscopic couple combines with the angular momentum of spin to make the wheel precess about a vertical axis, as we have seen.

It will also be understood why, as the spin of the wheel decreases, the rate of precession increases ; for, since the moment of inertia, I is constant and the torque T approximately so, the product $\omega.\Omega$ must be constant. Therefore as ω decreases Ω must increase.

We must take care not to forget that the axis of spin need not be horizontal : we have taken it so for convenience, but it may be in any direction whatever. In general it is convenient to think of the **three axes,** of **spin, torque,** and **precession,** as being mutually perpendicular : a gyroscopic couple, acting about an axis at right angles to the axle of spin, will produce precession about an axis approximately perpendicular to *both* the other axes (see Fig. 155).

STARTING PRECESSION. It must be noted that up to the present we have only considered the case of a body which is already precessing. The process of *starting* precession involves some further considerations, with which we must now deal.

Suppose that the wheel in Fig. 155 is simply spinning about its axle, so that it has no motion except rotation about that axle. If now a torque T is applied, gradually, to the ends of the axle AA, what will happen ?

First, the torque T will actually cause rotation through a very small angle about the axis BB. This rotation may be so small as to be almost imperceptible, but it *must* take place, as we shall see a little later. Consequently one end of the axle AA will rise slightly and the other end will fall an equal amount. The small angle ϕ thus turned through under the action of the torque T may be termed the **dip.**

The axle AA being now no longer horizontal, the vector aa_4 (Fig. 157) representing the *angular momentum of spin*, will also not be horizontal. Therefore the angular momentum of spin may be resolved into horizontal and vertical components. The horizontal component, which is much the larger, is represented by aa_1, and the vertical component, which is very small, by a_1a_4. This latter component is clearly angular momentum about a vertical axis.

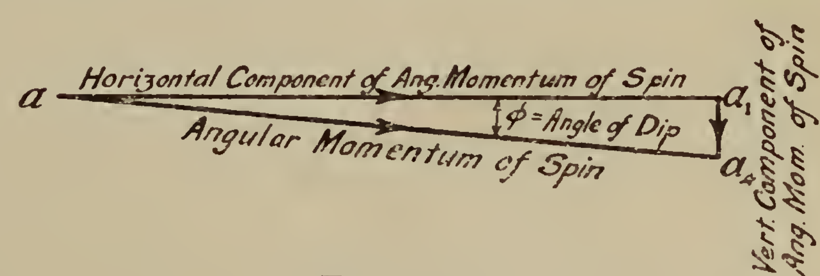


Fig. 157.

Now reference to page 257 will remind us that, by the Laws of Momentum, there can be no change in the angular momentum of a body unless the body is subjected to a torque which acts *in the same direction as the change of momentum.* The generation of angular momentum about a vertical axis, as indicated by the vector, a_1a_4, would seem therefore to be a contradiction of those laws, for there is here no torque acting about a vertical axis.

But immediately the torque T begins to turn the wheel about the axis BB, it begins also to generate angular momentum about that axis: this angular momentum about BB combines with the original angular momentum of spin (or rather its horizontal component), as shown in Fig. 156, and causes the wheel to precess about the vertical axis CC, as already explained. Now this precession about CC means that the wheel has angular

velocity and therefore angular momentum about CC.

The gyroscope has therefore *two* angular momenta about the vertical axis CC, viz.,

(i) the vertical component of the angular momentum of spin, and (ii) the angular momentum of precession.

Consider the directions of these two angular momenta, as indicated by Figs. 155 to 157.

The vertical component of the angular momentum of spin, if seen from above the wheel, would appear as *counterclockwise*, for its vector a_1a_4 is *downwards*.

The angular momentum of precession, if seen from above, would appear as *clockwise*, as shown by Fig. 155.

These two angular momenta about the same vertical axis CC are therefore in opposite directions and tend to balance each other: also the Laws of Momentum tell us that there can be no (resultant) angular momentum about this axis, since there is no torque about it.

It is evident, therefore, that these angular momenta must have zero resultant: that is, they must be of equal magnitude as well as in opposite directions. This gives us an explanation which agrees both with the observed experimental facts and with the fundamental laws of Mechanics.

WOBBLE. When a precessional torque (or gyroscopic couple) is first applied to the axle of a simply spinning body, the axle will *dip*, as just explained. But it will, in the first case, slightly overshoot the mark, dipping a little too far, especially if the torque is applied quickly. It will then rise again and will overshoot its correct position in the other direction. These oscillations of the axle will be accompanied by corresponding variations in the angular velocity of precession, but the effect of friction will be to damp them, and the system will soon settle down to steady dip and steady precession.

These oscillations are usually termed the "wobble" of the gyroscope, and are due to the fact that the torque can never be applied with *infinite* slowness. They are sometimes only evident as a slight tremor, the magnitude

of the effect depending on the conditions of the experiment. They may be shown by giving a sudden tap to the end of the axle AA, instead of applying a steady couple T. The more gradually the torque is applied, the less evident will be the wobble.

IMPEDING OR HURRYING PRECESSION. If we apply a torque to a precessing gyroscope, the direction of this extra torque being the same as the direction of the existing precession (i.e., about the axis CC), then experiment shows that the axle of spin will *rise* from its dipped position if the torque is applied so as to hurry the precession (i.e., if the extra torque has the same *sense* as the existing precession) ; while if the extra torque is applied so as to retard the precession (i.e., if the extra torque has the opposite sense to the existing precession), then the axle of spin will *dip* beyond its normal position.

The explanation of these phenomena should be clear to the student who has followed and understood the previous work. The extra torque, being applied in a direction at right angles to the direction of spin, causes a new precession about an axis perpendicular to both these axes, that is, about the axis BB.

The student may verify this experimentally by means of the bicycle wheel shown in Fig. 154. If he tries to turn about his own vertical axis more quickly or more slowly than he is impelled by the precession of the wheel, and holds the ball in such a way that the axle is always pointing straight in front of him but is free to increase or decrease its angle of dip, then he will find that when he turns quicker the axle rises and when he turns slower the axis dips.

KINETIC ENERGY OF GYROSCOPE. We have seen that the behaviour of the gyroscope is in accordance with the Laws of Momentum : we now have to show that it is in accordance with the Principle of the Conservation of Energy.

When the fly-wheel is simply spinning about its axle

AA, its kinetic energy is $\frac{1}{2}I\omega^2$ (as shown in Chapter 13, page 259).

When, in addition, the wheel is precessing under the action of a gyroscopic couple, this kinetic energy of spin will not be changed in any way, provided that the angular velocity of spin, ω, is maintained. The *dip* of the axle will make no difference, for kinetic energy is a scalar quantity : a change merely in the *direction* of spin leaves the energy unaffected.

The wheel will, however, have additional kinetic energy due to its angular velocity of precession. This additional energy will, of course, remain constant in magnitude so long as the angular velocity of precession remains constant.

From what source is this additional kinetic energy derived ?

The principles we have already studied give us the answer. Kinetic energy of rotation can only be imparted to a body by the action of a torque upon the body in such a way that the torque causes the body *to turn about the axis of the torque.* Those last words are important. The mere fact that a torque acts upon a body does not prove that work is being done by the torque, or that energy is being imparted to the body. The essential condition is that the torque shall impart to the body *motion about the axis of the torque.*

In the case we are considering, the only torque fulfilling this condition is the gyroscopic couple. This has acted upon the body and turned it through the angle of dip. Consequently the kinetic energy of precession must be equal to the work done on the body by the precessional torque, that is, it must be equal to the product :—

Precessional torque $\times$ Angle of dip (see page 260), i.e., it is equal to $T.\phi$.

It is therefore the work done by the gyroscopic couple in dipping the axle of spin that provides the kinetic energy of precession. We see, therefore, that gyroscopic motion is in accordance not only with the Laws

of Momentum, but also with the Principle of the Conservation of Energy.

ANALOGY WITH CENTRIPETAL FORCE. We have seen that, once precession has been started and is proceeding uniformly, the momentum generated by the torque T does not increase the total momentum possessed by the fly-wheel : it simply changes the *direction* of that momentum, just as the linear momentum generated by Centripetal Force does not increase the total linear momentum of a body travelling in a circle, but merely changes the *direction* of that momentum.

This analogy with Centripetal Force is interesting and useful. In the case of Centripetal Force we may say that the force is used *to rotate the linear momentum* of the body, and its magnitude is given by :—

Centripetal Force = Linear Momentum × Angular Velocity of Rotation

$$= \mathbf{M.v.}\omega.$$

In the case of Gyroscopic Torque we may say that the torque is used *to rotate the angular momentum* of the body, and its magnitude is given by :—

Gyroscopic Torque = Angular Momentum × Angular Velocity of Precession

$$= \mathbf{I}.\omega.\Omega.$$

It should be noticed that, in the case of uniform circular motion, Centripetal Force generates linear momentum but *does no work* ; and in the case of uniform gyroscopic motion, Precessional Torque generates angular momentum but *does no work.*

VEHICLE TRAVELLING ROUND A CURVE. The case of a railway engine or other vehicle travelling round a curve, gives an example of both Centripetal Force and Gyroscopic Torque.

The whole vehicle is changing its position and therefore has *linear momentum,* and this linear momentum necessitates *Centripetal Force* to change its direction and make it move along a curved path.

Simultaneously, the wheels of the vehicle are rotating about their axles and therefore have *angular momentum*, and this angular momentum necessitates *Gyroscopic Torque* to change its direction and make it move along a curved path, i.e., to precess.

It will be found that the direction of the gyroscopic torque required is such that, if the rails are both on the same level, as in a tramway, there will be a tendency for the vehicle to overturn outwards, i.e., in effect, gyroscopic action increases the tendency of the vehicle to overturn, which we studied in Chapter II. The student should verify this for himself, in accordance with the principles explained in the earlier part of this present chapter.

If the outer rail is given *superelevation* or *cant*, as explained in Chapter II, then clearly the axle of spin of the wheels of the vehicle is tilted through the angle of cant, and the gyroscopic torque provided by the increased upward pressure of the outer rail and the decreased upward pressure of the inner rail, will tend to make the wheels precess in the required direction. The cant, therefore, serves a double purpose : it provides both the centripetal force and the gyroscopic torque.

The magnitude of the gyroscopic torque on the wheels of a vehicle travelling round a curve will therefore be :—

$$T = I.\omega.\Omega,$$

where I = moment of inertia of wheels about their own axle,

ω = angular velocity of wheels about their own axle (i.e., angular velocity of spin),

and Ω = angular velocity of wheels round curve (i.e., angular velocity of precession).

But $$\omega = \frac{v}{r} \text{ and } \Omega = \frac{v}{R},$$

where v = linear speed of vehicle,

r = radius of wheels,

and R = radius of curved track.

Therefore $$T = I.\omega.\Omega$$
$$= I.\frac{v}{r}.\frac{v}{R}$$
$$= \frac{I.v^2}{r.R}.$$

DIMENSIONS. The dimensions of gyroscopic torque will be the same as those of any other torque, viz., $[M][L]^2[T]^{-2}$. This may be verified from the expression $T = I.\omega.\Omega$, for the dimensions of the moment of inertia I are $[M][L]^2$, and the dimensions of the angular velocities ω and Ω are each $[T]^{-1}$.

RULE FOR DIRECTION OF PRECESSION. The student who has worked carefully through this chapter should have no difficulty in determining the direction of precession in any given case from first principles, and that is the only safe way. It is convenient, however, to have a rule for the direction of precession which can be easily memorised and used as a check on results. The following is one of the simplest rules :—

Draw vectors, crossing each other at right angles, to represent the directions of **spin** and **torque**, as shown in Fig. 158, taking care that the arrow-heads denoting the sense of each vector are inserted correctly. Draw a curved arrow from the head of the spin vector to the head of the torque vector, as shown. Then this arrow will indicate the direction of **precession**.

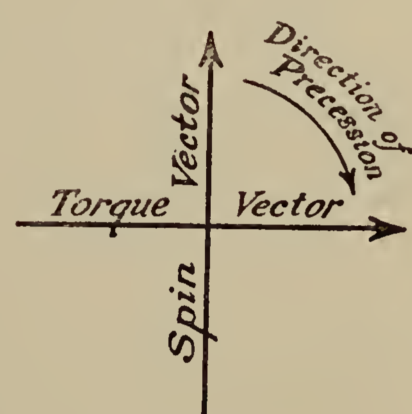


FIG. 158.

PEG-TOP AND HOOP. We can now see the explanation of the curious behaviour of the peg-top and the hoop, with the consideration of which we commenced the chapter. The overturning moment, due to the force of gravity and the reaction of the ground, acts on each of them in the same way as it acts on the bicycle

wheel in Fig. 154; that is, it causes *precession* instead of overturning.

We find that the peg-top, in addition to its spin about its own axis, tends to travel in circles on the ground. The faster it is spinning, the slower will be this motion of precession: as the spin diminishes the rate of precession increases, until at last the whole motion comes to an end with a convulsive roll.

So, also, the hoop, when it is acted upon by the gyroscopic couple consisting of its weight and the reaction of the ground, precesses by rolling along a curved path. It can readily be shown that the direction of this precession is in accordance with the rules we have given, and the student is advised to verify this for himself.

APPLICATIONS OF THE GYROSCOPE. In addition to the numerous cases where gyroscopic phenomena occur in ordinary mechanisms, such as the wheels of any vehicle, there are a number of instances where they have been deliberately invoked for useful purposes.

For example, the **mono-railways** of Brennan and of Schilowsky depend on gyroscopic action. The cars are fitted with high-speed gyroscopes ingeniously arranged to ensure that the upright position shall be maintained in all circumstances.

Similar devices have been invented for **steadying ships** at sea. Schlick successfully applied such an apparatus to a German torpedo-boat, reducing the total angle of rolling from 30 degrees to 1 degree, and good results have been obtained with other vessels since. The gyroscopes are kept spinning at high speed by means of electric motors.

The Whitehead **torpedo** is steered by means of a gyroscope. Any deviation from its intended path is remedied by the action of the gyroscope in controlling a rudder which brings the torpedo back to its correct direction.

One of the most important applications of all is the **gyroscopic compass.** This extremely ingenious inven-

tion by Dr. Anschütz has many advantages over the ordinary magnetic compass, providing a means of steering ships at sea which is far more accurate and reliable. As it does not depend in any way on magnetism, it is unaffected by changes in the earth's magnetic field, and by the presence of iron or steel. Consequently it can be used in situations where the magnetic compass would be unreliable.

The gyroscope employed for this purpose is run electrically at the high speed of 20,000 revolutions per minute, and is so arranged that the rotation of the earth causes the axle of the gyroscope always to point to *true* north. As *magnetic* north is variable, not being located at a constant point of the earth's surface, this is another item in favour of the gyroscopic compass compared with the magnetic ; especially as the setting of the axle is done automatically by the rotation of the earth.

The actual mechanisms of these various pieces of apparatus are too complicated for description here, but the student will see, even from these brief notes, the great and increasing importance of the principles involved.

SUMMARY OF CHAPTER 16

Gyroscopic phenomena are those which occur when it is attempted to turn the axis, about which a body is spinning, into a new position at an angle with its original position.

The **gyroscope** consists of a fly-wheel mounted on **gimbals** so that it is free to rotate about any one or more of three mutually perpendicular axes. The geometrical axis of the fly-wheel is termed its **axle** and the rotation of the wheel about this axle is termed its **spin.**

When a body is spinning about its axle and a torque is applied about an axis perpendicular to the axle, then the body rotates about an axis perpendicular to both the axle of spin and the axis of torque : this rotation is termed **precession.**

The torque applied in this way is termed the **precessional torque** or the **gyroscopic couple.**

When a body of moment of inertia I is spinning about its axle with angular velocity ω, the torque required to produce a precessional angular velocity Ω is equal to $I.\omega.\Omega$.

When precession is **first started** by a gyroscopic couple, the axle

is actually rotated about the axis of the couple through a small angle ϕ which is termed the angle of **dip.**

The **component of the angular momentum of spin** in the direction of the axis of precession, due to the dip, is equal and opposite to the **angular momentum of precession**: consequently there is no *resultant* angular momentum, about the axis of precession, generated by the gyroscopic couple.

A torque applied in such a direction as to **hurry the precession** will **reduce the dip,** while a torque applied so as to **retard the precession** will **increase the dip.**

The **kinetic energy of precession** is due to the work done by the precessional torque in turning the axle and wheel through the angle of dip.

Centripetal Force = Linear Momentum × Angular Velocity of Rotation.

Gyroscopic Torque = Angular Momentum × Angular Velocity of Precession.

Gyroscopic torque, like centripetal force, does no work.

In a vehicle travelling round a curve, the gyroscopic couple increases the tendency to overturn.

EXAMPLES XVI

(*For Hints on Working Examples, see page* 21.)

1. *A solid fly-wheel, of mass* 15 *pounds and radius* $3\frac{1}{2}$ *inches, is mounted on gimbals and given a spin of* 5,000 *revolutions per minute. If a perpendicular force of* 42 *pounds-weight is applied to one end of the axle, which is* $4\frac{1}{2}$ *inches in length, find the angular velocity of precession in revolutions per minute.*

Gyroscopic couple

$$= \text{force} \times \text{arm}$$
$$= 42 \text{ pounds-weight} \times 2\tfrac{1}{4} \text{ inches}$$
$$= \frac{42 \times 32{\cdot}2 \times 2\frac{1}{4}}{12} \text{ poundal-feet}$$
$$= 253{\cdot}8 \text{ poundal-feet.}$$

Moment of inertia of wheel

$$= \tfrac{1}{2} \text{ mass} \times \text{square of radius}$$
$$= \tfrac{1}{2} \times 15 \text{ pounds} \times (3\tfrac{1}{2} \text{ inches})^2$$
$$= \frac{\frac{1}{2} \times 15 \times 12\frac{1}{4}}{144} \text{ pound-foot-square}$$
$$= 0{\cdot}638 \text{ pound-foot-square.}$$

Angular velocity of spin

$$= 5{,}000 \text{ revolutions per minute}$$

$$= \frac{5{,}000 \times 2\pi}{60} \text{ radians per second}$$

$$= 524 \text{ radians per second.}$$

Angular velocity of precession

$$= \frac{\text{gyroscopic couple}}{\text{angular momentum of spin}}$$

$$= \frac{253{\cdot}8 \text{ poundal-feet}}{0{\cdot}638 \text{ pound-foot-square} \times 524 \text{ radians per second}}$$

$$= 0{\cdot}758 \text{ radian per second}$$

$$= 7{\cdot}24 \text{ revolutions per minute.}$$

2. *A pair of driving wheels on a railway engine have each a moment of inertia of* 16,000 *pound-feet-square and a diameter of* 5 *feet* 6 *inches. Find the gyroscopic couple acting on the wheels when the engine travels round a curve of* 15 *chains radius at a speed of* 55 *miles per hour.*

Angular velocity of spin of wheels

$$= \frac{\text{linear speed of engine}}{\text{radius of wheels}}$$

$$= \frac{88 \text{ feet per second} \times \frac{55}{60}}{2\frac{3}{4} \text{ feet}}$$

$$= 29{\cdot}33 \text{ radians per second.}$$

Angular velocity of precession

$$= \frac{\text{linear speed of engine}}{\text{radius of curve}}$$

$$= \frac{88 \text{ feet per second} \times \frac{55}{60}}{15 \times 66 \text{ feet}}$$

$$= {\cdot}0815 \text{ radian per second.}$$

Gyroscopic couple

$$= \text{moment of inertia of wheels} \times \text{angular velocity of spin} \times \text{angular velocity of precession}$$

$$= 2 \times 16{,}000 \text{ pound-feet-square} \times 29{\cdot}33 \text{ radians per second} \times {\cdot}0815 \text{ radian per second}$$

$$= 76{,}400 \text{ poundal-feet}$$

$$= \frac{76{,}400}{32{\cdot}2} \text{ pound-feet}$$

$$= 2{,}372 \text{ pound-feet.}$$

3. The moment of inertia of a gyroscope wheel is 10 pound-feet-square and the wheel is spinning with a velocity of 800 revolutions per minute. Find the torque, in pound-feet, required to give a precessional velocity of 15 revolutions per minute.

4. A fly-wheel, 3 feet in diameter and having a moment of inertia of 2,400 pound-feet-square, rotates with a rim speed of 100 feet per second. Find the rate of precession, in practical units, when the wheel is under the action of a gyroscopic couple of 180 pound-feet.

5. Determine the rate of spin of a gyroscope, given the following data: moment of inertia = 576 F.P.S. units; precessional velocity = $8\frac{1}{2}$ revolutions per minute; precessional torque = 200 pound-inches.

6. A traction engine has a cast-iron fly-wheel, of moment of inertia = $\frac{1}{2}$ ton-foot-square, diameter = 2 feet 6 inches, and rim speed = 72 feet per second. Calculate the magnitude of the gyroscopic torque acting on the bearings of the fly-wheel when the engine travels round a curve of 5 yards mean radius at a speed of 2 miles per hour.

7. Find the moment of inertia of a gyroscope wheel about its axle of spin, given the following data: rate of spin = 5,000 revolutions per minute, precessional torque = 330 pound-inches, rate of precession = 10 revolutions per minute.

8. Show that, on an aeroplane driven by a left-handed propeller, the gyroscopic action of the propeller will tend to dip the nose of the aeroplane when turning to the left, and to lift its nose when turning to the right.

9. A motor-cycle fly-wheel has a moment of inertia of 288 pound-inches-square, and a speed of 2,000 revolutions per minute when the cycle is travelling at a speed of 30 miles per hour. Find the gyroscopic torque due to travelling round a circle of 50 feet radius, assuming the cycle to remain upright. Neglect the gyroscopic effect on the cycle wheels.

10. In the previous example, what will be the gyroscopic effect if, in turning a corner, the cycle leans inwards towards the centre of the curved path?

11. Show that, if a paddle-steamer follows a zig-zag course, a tendency to *roll* will be produced, while if a turbine-driven single-screw steamer follows a similar course a tendency to *pitch* will result.

12. A torpedo-boat turns from due North to due East in 23 seconds. Find the gyroscopic couple on the propeller shaft, if the propeller makes 300 revolutions per minute, and its moment of inertia is 2,500 pound-feet-square.

13. Find the torque, in F.P.S. units, necessary to impart a preces-

sional velocity of 12 revolutions per minute to a gyroscope of which the moment of inertia is 88 pound-inches-square, and the rate of spin is 3,000 revolutions per minute.

14. Calculate the rate of precession of a gyroscope spinning at 2,200 revolutions per minute, given that the moment of inertia of the wheel is 25 million C.G.S. units, and the gyroscopic couple is 20 million C.G.S. units.

15. Determine the moment of inertia of a gyroscope wheel about its axle of spin, if a torque of 280 poundal-feet gives it a precessional velocity of 8 revolutions per minute when the rate of spin is 12,000 revolutions per minute.

CHAPTER 17: REVISION

WE have now completed our first consideration of the principles of Mechanics, and if we have really mastered the work we shall have laid a sound foundation on which to build up a more complete knowledge later. Two things, however, we must bear in mind: firstly, that there is still a great deal left for us to learn; and secondly, that unless we put our knowledge to the test of practical application as often, and in as many different ways, as possible, we cannot expect to have much real grip of it. It is easy to read a chapter of a text-book, or listen to a lecture, and to get a very fair idea of its meaning, but it is a very different matter to be able to apply such knowledge to practical problems. It must be impressed on the student that he cannot afford to rely on mere reading or listening to lectures. It is absolutely essential to work as many examples as possible, especially numerical examples. In this connexion attention may again be drawn to the practical hints given on pages 13 and 21.

We will now briefly review the work with which we have dealt, and remind ourselves of the most important of the principles which we have considered. It is not of the slightest use to pass on to more advanced work unless we thoroughly understand these fundamental principles and their relations one to another. As we have already suggested, it is a good practice to seize every opportunity of making *comparisons* between the various quantities which we encounter, and between the various definitions and rules, noting the points in which they resemble each other, and the points in which they differ. This will be found very helpful in fixing the important points on one's mind.

One of the first things that we noticed was that we can express all the quantities with which we have to deal

in Mechanics in terms of the three quantities **mass, length,** and **time,** which we therefore term our three fundamental quantities. This is a very important idea, because it shows us how we may find a means of determining the nature of the various quantities we have to consider, and how we may express them so that we can see readily to what extent they resemble each other and to what extent they differ.

Our practical way of doing this is to find what we term the physical **dimensions** of a quantity. These, it should be noted, are quite distinct from *geometrical* dimensions : they are simply an epitome of the kind of quantity it is, as shown by its relation to the three fundamental quantities. They help us to see something of the nature of quantities which would otherwise be rather difficult to comprehend. Another use for the idea of dimensions is as a means of checking the accuracy of formulæ and equations.

This same idea of three fundamental quantities enables us to find a basis for our systems of **units,** that is, systems of standard quantities with which to compare the magnitudes of other quantities. The units we choose for our three fundamental quantities become our three fundamental units, and all the other units of a system are directly derived from those three.

We noticed that units must be either **absolute** (i.e., constant in magnitude) or **variable.** In Mechanics we employ two systems of absolute units, the F.P.S. and C.G.S. systems. The units derived in the simplest possible way from the fundamental units are termed **systematic units** and there can only be one such for each quantity in each system. Other units may be employed for convenience in dealing with practical problems, but as far as possible systematic absolute units should be used in calculations in order to minimise the chance of error through inconsistent units being employed in the same piece of work.

An important classification is that of all quantities as either **vector** or **scalar.** The distinction between the

two classes should by now be perfectly clear: vector quantities have both magnitude and direction; scalar quantities have magnitude only. Obviously every quantity must belong to one category or the other.

Since vector quantities include the idea of direction as well as magnitude we cannot add or subtract them by the ordinary rules of algebra, as we can do when dealing with scalar quantities. Instead, we must employ the special methods of vector addition and vector subtraction which are embodied in the **Parallelogram Law.** Every student should make sure that he is thoroughly familiar with the methods of compounding together two or more vector quantities, and of resolving a vector quantity into components. It must never be forgotten that the Parallelogram Law applies to *all* vector quantities. Any doubts or uncertainties should be removed by careful revision and the working of examples: also the proof given in the Appendix should be studied.

The generalised idea of **moments** is deserving of careful consideration: we are constantly employing moments of one kind or another in Mechanics, and it is very helpful to have them arranged in our work in orderly fashion, rather than to take them piecemeal; for haphazard thinking leads to hazy notions, which it is our special aim to avoid. Such terms as *the first moment of momentum,* and *the second moment of mass,* should convey definite ideas to our minds from our general conception of a moment, even if we do not happen to be familiar with those particular quantities.

Of very special importance is a clear realisation of the fact, which we have emphasized repeatedly, that all motion is either **linear** or **angular.** No matter how complicated the motion of a body may appear, it can be shown to consist of these two kinds, **translation** and **rotation,** and of these only. The proportions may vary, one or the other may be absent, and one or both may be *variable,* but there is no third kind of motion.

It is interesting to compare the various quantities

with which we have to deal in considering these two kinds of motion, and to notice how, for every quantity we encounter in one kind of motion, there is a corresponding quantity in the other kind. In fact, we may say generally, that any law or statement which is true for one kind of motion, is equally true for the other if we substitute in it the appropriate equivalent quantities. The little table of comparison given in Chapter 13 (page 261) is worth careful study. We may note particularly that, wherever **mass** occurs in linear motion we must substitute for it the **second moment of mass** (moment of inertia) in angular motion, and that for **force** in linear motion we must substitute the **first moment of force** (torque) in angular motion.

We may also subdivide vector quantities into two groups or classes according to the kind of motion with which they are connected: those belonging to linear motion will then be termed **vector quantities of the first class**, while those belonging to angular motion will be termed **vector quantities of the second class.**

The principles governing the **equilibrium** of bodies under the action of forces should receive very careful attention. We can epitomise them by stating that, if the forces acting upon a body are in equilibrium, then they will not produce either *translation* or *rotation* of the body. So again we encounter our two kinds of motion! All the rules and conditions of equilibrium which can be laid down, however elaborate they may seem at first sight, are simply means of showing how we can give effect to this very simple general principle. They should, nevertheless, be studied with great care, for if the principle is simple, its application is apt to give much trouble. Here, again, the working of numerous examples is the only sure road to success, and even this will lose much of its value unless it is accompanied by careful cogitation.

In dealing with **velocity** and **acceleration** one of the most essential points to remember is that they are vector quantities; therefore they are only constant so long

as both their magnitude *and their direction* are constant. This may seem rather a trite statement at this stage of our studies, but it is one which may none the less be overlooked if we are not very careful. In this connexion it is useful to recall some statements which we made in the chapter on rotation, when dealing with normal acceleration (see page 128):—

(*a*) An acceleration which is in the *same* direction as the direction of motion of a body, will produce a change of *speed* only.

(*b*) An acceleration which is in a direction *perpendicular* to the direction of motion of a body, will produce a change of *direction* only.

(*c*) An acceleration which is in any other direction will produce a change of both speed and direction.

The study of *circular motion* will help to emphasize these points, and is of considerable importance for its own sake also.

One of the most vital parts of our subject is that which deals with **momentum,** both linear and angular. Here the objects of our special attention should be the **Laws of Momentum,** which are the equivalent of Newton's Laws of Motion. The laws of momentum should be studied in their application to motion of rotation as well as motion of translation, and the two cases should be the object of careful comparison. These laws when once thoroughly understood can hardly be forgotten, for they are readily accepted by the mind as intelligible if not axiomatic. Once more it must be emphasized that the working of large numbers of examples is the best way of fixing principles in one's memory and understanding.

The **Principle of the Conservation of Energy** is another of the outstanding laws with which we must familiarise ourselves. It is indeed the foundation of a very large part of physical science in general, apart from its very great importance in Mechanics. A thorough grip of the principle, and the ability, acquired by constant practice, to apply it readily, will enable us to

solve many problems of very varied character. It is necessary, therefore, not merely to *know* the law and to be able to repeat it, but also really to comprehend its significance and to be able to see when and how to use it. This again demands that the student shall take a keen interest in his work and shall make a practice of pondering over each new aspect of the work as he comes across it. Every fresh problem should be considered from the point of view of the *why* as well as the *how*.

In dealing with kinetic energy we should again apply our method of comparison. We may with advantage study the points of resemblance and the points of difference of energy and momentum, as already indicated on pages 236–7 and 261. We may also compare kinetic energy of translation with kinetic energy of rotation.

We have now passed very briefly in review some of the fundamental principles of Mechanics: we might say *the* fundamental principles, for there are no others of equal importance. If, after reaching this point, the student finds that he is still uncertain about any of these essential parts of the subject, then he is strongly advised to turn back to those parts as to which he is in doubt, and to study them again and again until he is thoroughly clear about them, and feels assured that he can apply them to any simple problems without hesitation or difficulty. It is not of the slightest use to go on to more advanced work until the elementary part of the subject is thoroughly mastered.

We must now glance rapidly at some of the other parts of the work which we have considered in previous chapters. These parts, although not of the outstanding importance of those already noticed in this summary, are only second to them in importance, and no student can claim an adequate knowledge of elementary Mechanics if he does not know something of them.

The study of the internal forces in the materials of which bodies are composed, and of the effects which such forces produce on the size and shape of the bodies concerned, is of great use in engineering and of considerable

interest. It is, of course, the study of **stress** and **strain**, and that property of bodies which we term **elasticity.** Here **Hooke's Law** is the most important principle to be mastered, and in studying this we shall familiarise ourselves with the ideas of the **elastic limit** and of the **moduli of elasticity.**

We must also acquire some knowledge of the special forces which we term **gravity** and **friction.** In dealing with the former we must take particular care to be absolutely clear as to the distinction between **mass** and **weight.** It would hardly seem possible to confuse two quantities so entirely different, if one did not know that it is frequently done: to make so crude a mistake argues a great lack of appreciation of the basic ideas of Mechanics.

In considering friction, careful performance of the simple experiments mentioned in Chapter 9 will be found very useful and instructive. The Laws of Friction for plane, dry surfaces should be studied attentively, and the significance of the **coefficient of friction** grasped.

Two special cases of equilibrium deserve a word to themselves. The first is the case of three forces in equilibrium meeting at a point: this introduces the **Triangle of Forces,** a theorem which is not only of interest from an academic point of view, but is also of great value in practical work, for it is the basis of the **stress diagram** so widely used by engineers in the design of framed structures. An example of this use has been worked out in detail at the end of Chapter 10, and will be found of interest.

The other case to which we wish to draw attention is that of three forces acting upon a rigid body and keeping it in equilibrium. Here we have a theorem which tells us that in such a case the three forces must either all meet in one point or else all be parallel. It will be found that in this simple theorem lies the key to the easy solution of a considerable number of problems in equilibrium: it is therefore one which the student

will be well advised to study carefully: needless to say, this can only be done adequately by combining practice with theory and working many numerical examples. Laws are not of much service to us unless we know, by experience, how to apply them.

Three chapters have been devoted to special cases of motion, under the headings of **centripetal force, periodic motion,** and **gyroscopic motion.** These topics are interesting in themselves and of practical importance to the engineer.

Two other chapters have been devoted to themes which are somewhat mathematical in character, but are of such importance in Mechanics that we may well consider them as part of our subject. These are, of course, the chapters on **centre of gravity** and **moment of inertia.** The latter involves the use of the integral calculus, so it is possible that some students may have to defer its study to a later date: if so, then it should be impressed on such students that it will be well worth their taking the trouble to acquire an elementary knowledge of the differential and integral calculus as soon as possible. It is necessary that such a knowledge should be *thorough*, but it is by no means necessary that it should be very *wide*: the scientist and the engineer use the calculus very frequently, but in the great majority of instances they only use the very simple parts of it. In this connexion, Professor Silvanus P. Thompson's little book, "Calculus Made Easy," can be strongly recommended.

EXAMPLES XVII

(*For Hints on Working Examples, see page* 21.)

1. *The momentum of a body is* 124 *F.P.S. units and its speed is* 150 *miles per hour. Determine, from first principles, its kinetic energy. If its volume is* 1·96 *cubic inch, what is its density?*

By definition, the momentum of a moving body is equal to the product of its mass and its velocity.

$$\therefore \text{ Mass of body} = \frac{\text{momentum}}{\text{velocity}}$$
$$= \frac{124 \text{ F.P.S. units}}{(150 \times \frac{22}{15}) \text{ feet per second}}$$
$$= \cdot 564 \text{ pound.}$$
$$\text{Density} = \frac{\text{mass}}{\text{volume}}$$
$$= \frac{\cdot 564 \text{ pound}}{1 \cdot 96 \text{ cubic inch}}$$
$$= \cdot 288 \text{ pound per cubic inch.}$$

If the momentum of the body were destroyed in time t seconds, then the force it would exert would be :—

$$\text{Force} = \frac{\text{momentum}}{\text{time}}$$
$$= \frac{124}{t} \text{ poundals.}$$

In this time it would travel a distance

$$= \text{average velocity} \times \text{time}$$
$$= \tfrac{1}{2} \times 220 \text{ feet per second} \times t$$
$$= 110.t \text{ feet.}$$

The work done, and hence the kinetic energy of the moving body, would be equal to the product of force and distance, i.e.,

$$\text{Kinetic energy} = \frac{124}{t} \times 110.t = 13{,}640 \text{ foot-poundals.}$$

2. *In terms of the fundamental units of mass* [M], *length* [L] *and time* [T], *give the dimensions of Momentum, Force, Kinetic Energy, Work Done, Moment of a Force, and Power. Which of these are Vector Quantities and which Scalar? Show that the equation* $T = 2\pi\sqrt{\dfrac{h}{l^2 + k^2}}\, g$, *where* T *= a time,* g *= acceleration due to gravity, and* l, h *and* k *are lengths, cannot be correct, and arrange the terms* h, $l^2 + k^2$, g *so that the equation is correct.* (Univ. Lond. Int. Eng. 1920.)

Momentum : $[M][L][T]^{-1}$: vector quantity.
Force : $[M][L][T]^{-2}$: vector quantity.
Kinetic Energy : $[M][L]^2[T]^{-2}$: scalar quantity.
Work Done : $[M][L]^2[T]^{-2}$: scalar quantity.
Moment of a Force : $[M][L]^2[T]^{-2}$: vector quantity.
Power : $[M][L]^2[T]^{-3}$: scalar quantity.

Taking the dimensions of both sides of the given equation, we have, as it stands :—

$$[\mathrm{T}] = \sqrt{\frac{[\mathrm{L}]}{[\mathrm{L}]^2 (+ [\mathrm{L}]^2)}\,[\mathrm{L}][\mathrm{T}]^{-2}}$$

$$= \left\{\frac{[\mathrm{L}]^2[\mathrm{T}]^{-2}}{[\mathrm{L}]^2}\right\}^{\frac{1}{2}}$$

$$= [\mathrm{T}]^{-1}, \qquad \text{which is absurd.}$$

To correct the equation, the dimensions of time on the right-hand side must be made to equal [T] : this can only be done by dividing by g instead of multiplying. Having done this, the dimensions of length will be incorrect : they can only be corrected by interchanging the terms h and $(l^2 + k^2)$. The corrected equation will then read

$$\mathrm{T} = 2\,\pi\sqrt{\frac{l^2 + k^2}{h.g}}.$$

The dimensions on the right-hand side are now $\sqrt{[\mathrm{L}]^2/[\mathrm{L}]^2[\mathrm{T}]^{-2}}$, that is [T], which is correct.

3. *A spring which compresses* 1 *inch per* 100 *pounds load acts as a buffer and is compressed initially* 1 *inch. What would be the final compression and the force on the spring if it absorbs as a buffer* 120 *foot-pounds of energy?* (Assoc.M.Inst.C.E. 1920.)

Let x = distance spring is compressed as a buffer, in inches.

Then final force on spring $= 100(x + 1)$ pounds-weight.

Average force exerted as buffer $= \frac{1}{2}\{100 + 100(x + 1)\}$ pounds-weight

$= (50.x + 100)$ pounds-weight.

Energy absorbed as a buffer

$= \text{force} \times \text{distance}$

$= (50.x + 100) \times x$

$= (50.x^2 + 100.x)$ inch-pounds.

But this is equal to 120 foot-pounds : therefore :—

$$50.x^2 + 100.x = 120 \times 12,$$

whence $x^2 + 2x - 28{\cdot}8 = 0$

and $x = \frac{1}{2}(-2 \pm \sqrt{4 + 115{\cdot}2})$

$= 4{\cdot}455$ inches.

Therefore final compression

$= x + 1$

$= 5{\cdot}455$ inches.

Final force on spring $= 100(x + 1)$

$= 545{\cdot}5$ pounds-weight.

4. *A uniform cube rests with a face touching the highest point of a fixed rough sphere. Prove from fundamental principles that the equilibrium is stable if the edge of the cube is less than the diameter of the sphere.*
(Univ. Lond. Int. Eng. 1919.)

Let A be the highest point of the sphere, and let A′ be the point of the cube which is in contact with A. Clearly A′ must be the centroid of the lowest face of the cube, and the latter must be horizontal.

Let $2l$ = edge of cube.
r = radius of sphere.

Suppose the cube to be slightly displaced by rolling on the sphere without slipping, and let B be the new point of contact. Then if AB $= x$, the angle of displacement will be

$$d\theta = \frac{x}{r} \text{ radians.}$$

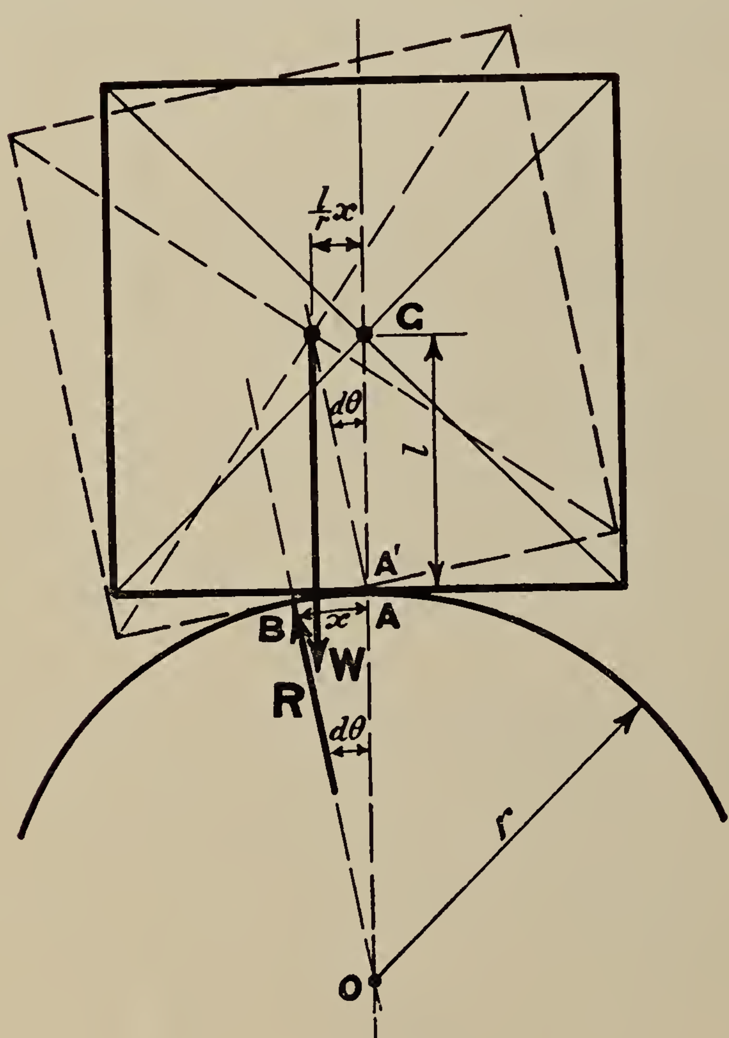


FIG. 159.

The reaction of the sphere on the cube will act through the point of contact B.

The centre of gravity of the cube will be displaced a distance equal to $l.d\theta$, that is, a distance $l.x/r$. The line of action of the weight of the cube will therefore be displaced to the same extent. The equilibrium of the cube will be stable if the couple formed by the weight of the cube, and the vertical component of the reaction of the sphere on the cube, tends to restore the cube to its original position. The condition for this is that the line of action of the weight of the body shall not be displaced further than the point of contact is displaced, i.e., that the new line of action of the weight shall pass between A and B.

This requires that $\frac{l}{r}x$ shall be less than x, i.e., that l shall be less than r; so that the equilibrium is stable if the edge of the cube is less than the diameter of the sphere. (*Q.E.D.*)

5. Plot a curve to show the relation between the speed of rotation (in revolutions per minute) of a conical pendulum, and its height h, in inches. What are the highest and lowest speeds at which such a pendulum can rotate when used as a simple governor, if the arms are to make an angle of 45 degrees with the spindle, and their length is not to exceed 18 inches, nor be less than 6 inches ?

6. Find the work done in dragging a mass of 18 pounds up a plane inclined at 35 degrees to the horizontal, and 14 feet 6 inches in length, if the coefficient of friction is ·23 and the force acts in a horizontal direction.

7. Find the superelevation which must be given to the outer rail of a metre-gauge railway, on a curve of 450 metres radius, so that a train may travel round it at 50 kilometres per hour without side-thrust between rails and wheels.

8. The weight on the driving wheels of a locomotive is 43 tons. If the coefficient of friction between the wheels and the rails is ·095, and the tractive resistance is $(\frac{1}{4}V + 2\frac{1}{2})$ pounds-weight per ton, what is the maximum speed V, in miles per hour, attainable on the level when hauling a train load of 600 tons mass, including the locomotive ? What will be the acceleration of the train when travelling at 25 miles per hour, full steam ahead ?

9. Find the moment of inertia of a steel cube of 15 centimetres edge, about one edge. Specific gravity of steel = 7·85.

10. Two smooth heavy spheres, of different radii, are attached to the ends of a fine string ; the string passes over a smooth peg and the spheres rest in contact. Prove that the weights of the spheres are inversely as the distances of their centres from the peg. (*Bd. of Educ. Theor. Mechs., Lower,* 1914.)

11. From a consideration of dimensions only, show that the periodic time of a simple pendulum is proportionate to $\sqrt{l/g}$ (where l is the length of the pendulum), and is independent of the mass of the bob.

12. What stress in kilogrammes-weight per square centimetre is equal to a stress of $7\frac{1}{2}$ tons-weight per square inch ?

13. A uniform stick of length l is placed in the rough ring of an umbrella stand at a height h from the ground. It rests also on a smooth floor. Show that equilibrium is impossible unless the stick is vertical, if the coefficient of friction is less than $h/\sqrt{(l^2 - h^2)}$. (*Univ. Lond. Int. Eng.* 1919.)

14. Find the radius of gyration of a hollow sphere, of inside radius r and outside radius R, about a diameter.

15. A body moves along a circular path, of 40 centimetres radius,

with uniform angular acceleration of one C.G.S. unit. Find its resultant acceleration when it has just completed one revolution from rest.

16. A plank AB, 11 feet in length, and weighing one hundredweight, rests horizontally and at right-angles across a wall 18 inches in width. The end A is 4 feet from the nearer face of the wall, and a load of 2 hundredweight is placed on the plank over the middle of the wall. How far along the plank towards either end can a boy, weighing 6 stone, walk without causing the plank to overturn ?

17. A steel column, 20·6 square inches in cross-sectional area and 24 feet in length, carries a load of 56 tons-weight. Taking E as 13,000 tons-weight per square inch, calculate how much the column is shortened by the load.

18. A bullet weighing one ounce is fired into a block of wood weighing 5 pounds which is initially at rest. If the velocity of the bullet and block together after impact is 15 miles per hour, determine the velocity with which the bullet strikes the block, and the loss of energy due to the impact.

19. Find the length of the simple pendulum which will have a periodic time of $3\frac{1}{4}$ seconds, if the value of g is 982 centimetres per second per second.

20. Determine the force required to impart a velocity of a mile a minute to a motor-car, weighing 2,560 pounds, in $2\frac{1}{4}$ minutes from rest. What will be the gain of energy in the last minute of this period ?

21. The angular velocity of the driving wheels of a railway locomotive is 32 radians per second when the train is travelling at 64 miles per hour. Find the diameter of the wheels.

22. The mass of a rotating body is ·33 ton and its radius of gyration is 4 feet $7\frac{3}{4}$ inches. If it is acted upon by a constant tangential force of 32 pounds-weight at a distance from the axis of rotation of 5 feet 3 inches, find its kinetic energy and angular momentum after $1\frac{1}{2}$ minutes from rest.

23. A body, of 2 pounds mass, is whirled in a vertical circle at the end of a string 3 feet in length. Find the least speed at the lowest point, so that the body may complete the circle, and the tension in the string in the two positions in which it makes an angle of 45 degrees from the vertical.

APPENDIX

PROOF OF THE PARALLELOGRAM LAW

We will take the proof for the Parallelogram of Forces, as that is the most difficult case of the Parallelogram Law. The student will find that he can readily adapt the proof to other cases, such as the Parallelogram of Velocities, the Parallelogram of Accelerations, and the Parallelogram of Momenta.

Let two forces, P and Q, be represented in magnitude and direction by the vectors OA and OB (Fig. 160), drawn from the same point O.

Suppose that both the forces act upon a particle of mass m, for a time t.

If P were the only force acting upon the particle, then the particle would receive an acceleration a_1, equal to P/m, in the direction OA, and, therefore, in the time t its velocity in that direction would become $v_1 = a_1.t = P.t/m$. The distance travelled in the same direction would be $s_1 = \frac{1}{2}.v_1.t = P.t^2/2.m$, that is, a distance proportional to P and in the same direction.

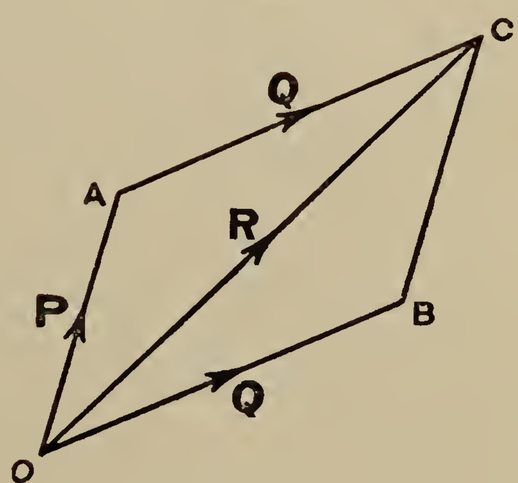


FIG. 160.

If Q were the only force acting, then the particle would receive an acceleration a_2, equal to Q/m, in the direction OB, and therefore in the time t its velocity in that direction would become $v_2 = a_2.t = Q.t/m$. The distance travelled in the same direction would be $s_2 = \frac{1}{2}.v_2.t = Q.t^2/2.m$, that is, a distance proportional to Q and in the same direction.

Now if OA be taken to represent the distance travelled under the action of the force P, then AC, which is equal to OB, will represent the distance travelled in the same time under the action of the force Q.

Therefore OC will represent, to the same scale, the distance travelled under the combined action of the two forces P and Q. But we have just shown that the distance travelled is always proportional to the force acting and is in the same direction. Therefore OC = R, must represent the resultant of P and Q both in magnitude and direction.

PROOF OF VARIGNON'S THEOREM

Let P and Q (Fig. 161) be forces acting in one plane upon a rigid body, and let their lines of action intersect in X. The line of action

of the resultant R of these forces must then, by the Parallelogram of Forces, also pass through the same point X.

Consider the moments of the forces about any point O in their plane. The moment of P about O is equal to P × OA, where OA is the perpendicular distance of the line of action of P from O. Similarly, the moment of Q about O is equal to Q × OB, and the moment of R about O is equal to R×OC.

Join OX and draw DXE perpendicular to OX. Let the angle AXE = α, the angle BXD = β, and the angle CXE = γ.

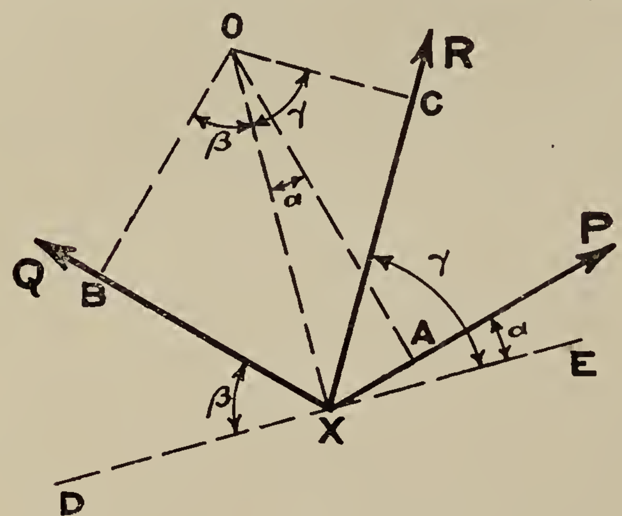


FIG. 161.

The resultant moment, about O, of the forces P and Q
= sum of moments of separate forces about O
= (P × OA) − (Q × OB)
= (P × OX.cos α) − (Q × OX.cos β)
= OX(P.cos α − Q.cos β)
= OX(sum of resolved parts of P and Q in the direction DXE)
= OX(resolved part of R in the direction DXE) (since R is the resultant of P and Q)
= OX(R.cos γ)
= R × OC
= moment of resultant force about O:

which proves the theorem.

The moments of P and Q have been taken as of opposite sign, but the same proof will apply if they are of the same sign as each other. Clearly also the proof can be extended to any number of coplanar forces.

GRAPHICAL KINEMATICS

It is impossible in the space at our disposal to give anything like a complete account of the application of graphical methods to the solution of problems connected with velocity and acceleration. All that we can do is to indicate very briefly a few of the most important results. For further information the student is recommended to consult Landon's "Elementary Dynamics."

(*a*) The **velocity** of a moving body is defined as its *rate of change of position.*

If we plot, on squared paper, the **position** of a moving body against the **time** at which it occupies that position, then we obtain a POSITION-TIME CURVE, as shown in Fig. 162.

It can be shown that *the slope of the tangent to the curve, at any*

point A, gives the rate of change of position, i.e., the velocity, corresponding to the time and position represented by that point.

Then if s is the position of the moving body at time t, the corresponding velocity

$$= v = \tan\ \theta$$
$$= \frac{ds}{dt}.$$

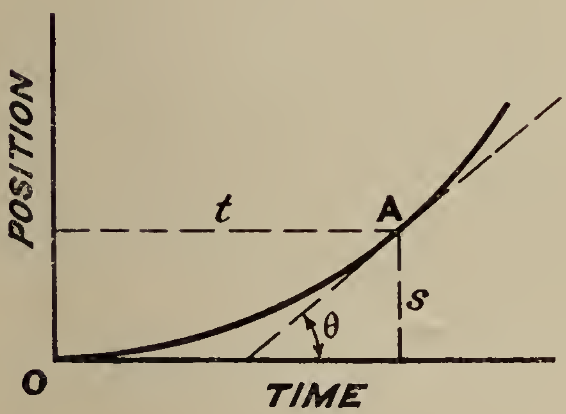


FIG. 162.

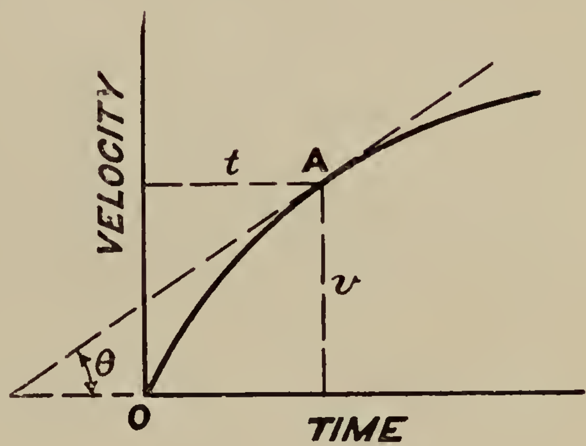


FIG. 163.

(*b*) The **acceleration** of a moving body is defined as its *rate of change of velocity.*

If we plot, on squared paper, the **velocity** of a moving body against the **time** at which it has that velocity, then we obtain a VELOCITY-TIME CURVE, as shown in Fig. 163.

It can be proved that *the slope of the tangent to the curve, at any point A, gives the rate of change of velocity, i.e., the acceleration,* corresponding to the time and velocity represented by that point.

Then if v is the velocity of the moving body at time t, the corresponding acceleration

$$= a = \tan \theta$$
$$= \frac{dv}{dt}$$
$$= \frac{d^2s}{dt^2}.$$

Also, *the area under the velocity-time curve, or under any part of it, gives the* **change of position** *in the corresponding time.*

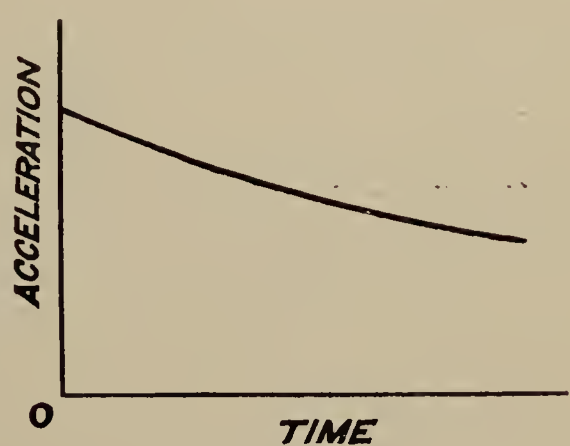


FIG. 164.

(*c*) If we plot, on squared paper, the **acceleration** of a moving body against the **time** at which it has that acceleration, then we obtain an ACCELERATION-TIME CURVE, as shown in Fig. 164.

It can be proved that *the area under the acceleration-time curve,*

or under any part of it, gives the **change of velocity** *in the corresponding time.*

To obtain the *velocity-time* curve from the *position-time* curve, or the *acceleration-time* curve from the *velocity-time* curve, we find the values of the tangents of the angles of slope of tangents to the primitive curve (that is, the curve which is given to us initially), at various points, and plot those values against the corresponding times.

To obtain the *velocity-time* curve from the *acceleration-time* curve, or the *position-time* curve from the *velocity-time* curve, we find the values of the area under the primitive curve from the origin up to various points, and plot those values against the corresponding times.

It may be noted that, in Fig. 162, while tan θ, or ds/dt, gives the **actual velocity** at A, s/t gives the **mean velocity** from O to A. Similarly, in Fig. 163, while tan θ, or dv/dt, gives the **actual acceleration** at A, v/t gives the **mean acceleration** from O to A.

MECHANICAL QUANTITIES IN TERMS OF THE CALCULUS

Many of the quantities used in Mechanics can be conveniently expressed in the notation of the calculus. Students who are familiar with that notation may find it instructive to study the following list of a few such quantities, and to add to it other similar expressions.

Quantity.	Symbol.	Expression as Differential Coefficient.	Expression as Integral.
Position	s	—	$\int v.dt$
Velocity	v	$\frac{ds}{dt}$	$\int a.dt$
Acceleration	a	$\frac{dv}{dt} = \frac{d^2s}{dt^2}$	—
Momentum	M.v	$\mathrm{M}.\frac{ds}{dt}$	$\int \mathrm{P}.dt$
Force	P	$\mathrm{M}.\frac{dv}{dt} = \mathrm{M}.\frac{d^2s}{dt^2}$	—
Angular position . . .	θ	—	$\int \omega.dt$
Angular velocity . . .	ω	$\frac{d\theta}{dt}$	$\int a.dt$
Angular acceleration . .	a	$\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$	—
Angular momentum . .	I.ω	$\mathrm{I}.\frac{d\theta}{dt}$	$\int \mathrm{T}.dt$
Torque	T	$\mathrm{I}.\frac{d\omega}{dt} = \mathrm{I}.\frac{d^2\theta}{dt^2}$	—
Work (or energy) . . .	P.s	—	$\int \mathrm{P}.ds$
ditto	T.θ	—	$\int \mathrm{T}.d\theta$

LIST OF DEFINITIONS

Absolute units are those which always have the same value.

Acceleration is rate of change of velocity.

Active forces are those which can originate motion.

Amplitude, of a simple harmonic motion, is the distance from the middle point of the path to either end of the path.

Angle of friction is the angle whose tangent is equal to the coefficient of friction.

Angular acceleration is rate of change of angular velocity.

Angular momentum is the quantity of angular motion possessed by a body, and is equal to the product of the moment of inertia of the body, about its axis of rotation, and its angular velocity.

Angular velocity is rate of rotation.

Axle of a gyroscope is the geometrical axis of its fly-wheel.

Beam is a bar subjected to bending.

Cant is the lateral inclination given to a road or railway track in order to avoid side-thrust on the wheels.

Centre of gravity of a body is the point of application of its weight.

Centre of mass of a body is the point at which the whole of its mass may be considered to be concentrated.

Centre of oscillation is the point in a compound pendulum which corresponds with the position of the bob in the simple equivalent pendulum.

Centre of parallel forces is the point of application of their resultant.

Centre of suspension is the point at which the axis of suspension of a compound pendulum cuts the vertical plane in which the centre of gravity swings to and fro.

Centrifugal force is the equal and opposite reaction to centripetal force, and is exerted by the moving body on the means of constraint.

Centripetal force is the force required to compel a body to move in a curved path.

Centroid of an area is its effective centre.

Coefficient of friction is the constant ratio, for a given pair of surfaces, of the force of friction to the normal pressure between them.

Components of a force in any given directions are those forces which would together have the same effect as the original force.

Composition of forces is the process of adding forces together by means of the Parallelogram Law.

Compound pendulum is a body of any shape suspended at a horizontal axis and allowed to swing to and fro through a small angle under the action of the force of gravity.

Compression is the state of a body which is subject to external pushes, tending to shorten it in the direction of their action.

Compressive stress is the pair of internal forces in the material of a body which is in a state of compression.

Conical pendulum is a small heavy body tied by a light string to a fixed point and caused to rotate in a horizontal circle.

Couple is a pair of equal, unlike, parallel forces not in the same straight line.

Density is mass per unit volume.

Derived units are those obtained in any way from the fundamental units.

Dimensions of a physical quantity are the three factors composing it, where each factor is one of the three fundamental quantities, raised to some integral power which may be either positive or negative or may be zero.

Direct stress is the stress produced in a body by the action of a pair of equal and opposite forces which are in the same straight line.

Displacement is change of position.

Displacement in simple harmonic motion is the distance of the moving body at any instant from its middle position.

Dyne is that force which will give an acceleration of one centimetre per second per second to a mass of one gramme.

Efficiency of a machine is the ratio of the useful work obtained from it to the total work put into it.

Elasticity is the property of a body by virtue of which it loses strain when stress is removed.

Elastic limit is the maximum stress which can be applied to a piece of material without causing permanent distortion.

Energy is capacity for doing work.

Equilibrant is the force which must be added to a system of forces in order that they may be in equilibrium: it is equal and opposite to the resultant of the original forces.

Equilibrium is the state of a body which is not subject to acceleration, i.e., which is either at rest or in uniform motion.

Erg is the work done by a force of one dyne acting through a distance of one centimetre.

Factor of safety is the ratio of the breaking stress on a piece of material to the actual stress applied to it.

Foot-pound is the work done when a force of one pound-weight acts through a distance of one foot.

Foot-poundal is the work done when a force of one poundal acts through a distance of one foot.

Force is that which produces, or tends to produce, motion or change of motion in the matter upon which it acts.

Force ratio of a machine is its *mechanical advantage* (*q.v.*).

Friction is the resistance to the sliding of one body over another.

Fulcrum of a lever is its fixed point or axis.

Fundamental quantities are Mass, Length, and Time.

Fundamental units are arbitrarily chosen absolute units of Mass, Length, and Time.

Gimbals are the pivoted frames employed to carry the fly-wheel of a gyroscope.

Gravitation units are those depending on the magnitude of the force of gravity, and varying with its variations.

Gravity is the force of attraction exerted by the earth upon all material bodies.

Gyroscope is a fly-wheel mounted on gimbals so that it is free to rotate in any direction.

Gyroscopic phenomena are those which occur when it is attempted to turn the axis of a spinning body into a new position at an angle with its original position.

Horse-power is a rate of doing work equal to 33,000 foot-pounds per minute.

Impulse is a sudden change of momentum.

Impulsive force is a very great force acting for a very short time.

Inertia is the property of a body by reason of which it requires force to set it in motion or to vary its motion.

Joule is a unit of work equal to ten million ergs.

Kilowatt is a rate of doing work equal to one thousand joules per second.

Kinematics is that part of Mechanics which is concerned with the properties of motion itself, without considering the forces involved.

Kinetic energy is the capacity of a body for doing work due to its motion.

Kinetic energy of rotation is the energy possessed by a body by reason of the speed with which it is rotating.

Kinetic energy of translation is the energy possessed by a body by reason of the speed with which it is moving from one place to another.

Kinetics is that part of Mechanics which is concerned with the forces producing motion.

Limiting friction is the maximum value of the force of friction for a given pair of surfaces.

Linear acceleration is rate of change of linear velocity.

Linear momentum is the quantity of linear motion possessed by a body, and is equal to the product of its mass and its linear velocity.

Linear motion is motion from one place to another.

Linear velocity is rate of translation.

Machine is a piece of apparatus for changing the direction or magnitude of a force.

Mass is the quantity of matter in a body.

Mechanical advantage is the ratio of load to effort in a machine.

Modulus of elasticity is the ratio of stress to strain for a particular material.

Modulus of rigidity is the ratio of shear stress to shear strain for a particular material.

Moment (or First Moment) of any physical quantity about a point or axis is the product of the quantity and its effective distance from that point or axis.

Moment of inertia of a body about a given axis is the sum of the products of the mass of each particle of the body into the square of its distance from the given axis : another name for it is Second Moment of Mass (*q.v.*).

Moment of momentum (or Angular Momentum) of a rotating body is equal to the product of the moment of inertia of the body about the axis of rotation and the angular velocity of the body.

Momentum (or Linear Momentum) of a body is the quantity of linear motion it possesses, and is measured by the product of the mass of the body and its linear velocity.

Normal acceleration is an acceleration which does not produce a change of speed but only a change of direction of motion, its direction being perpendicular to the direction of motion at any instant.

Overturning moment is the moment of the couple which tends to upset a vehicle or other body from its normal position.

Passive forces are those which cannot originate motion, being merely called into play by active forces.

Pendulum is a body which swings to and fro through a small angle under the action of gravity.

Periodic motion is motion of which every part recurs regularly.

Periodic time of simple harmonic motion is the time occupied in moving from one end of the path to the other and back again.

Polar moment of inertia of a lamina is its moment of inertia about an axis perpendicular to its plane.

Potential energy is the capacity of a body for doing work by reason of its position and the action of the force of gravity upon it.

Poundal is that force which will give an acceleration of one foot per second per second to a mass of one pound.

Power is rate of doing work.

Precession is the rotation produced by a gyroscopic couple.

Radius of gyration is the effective distance of the mass of a solid body, or the area of a plane figure, from a given axis.

Reaction is the passive force called into play by an active force.

Relative velocity is the rate of displacement of one body with respect to another.

Resolution of forces is the process of splitting forces up into components.

Resolved parts of a force are components of a force in directions which are mutually perpendicular.

Resultant of a system of forces is that force which would have the same effect as the whole of the original forces together.

Righting moment is the moment of the couple which tends to restore a vehicle or other body to its normal position.

Scalar quantities are those which have magnitude but not direction.

Second moment of any physical quantity about a given axis is the product of the quantity and the square of its effective distance from that axis.

Sense of a vector indicates in which of the two possible (opposite) directions it is to be taken.

Shear is the state of a body which is subject to external couples, tending to distort its shape.

Shear stress is the pair of internal couples in the material of a body which is in a state of shear.

Simple equivalent pendulum is the simple pendulum which would have the same periodic time as a given compound pendulum.

Simple harmonic motion is motion backwards and forwards along a path, such that the acceleration of the moving body is always directed towards some fixed point in the path and is proportional to the distance from that point, measured along the path.

Simple pendulum is a small heavy body tied to a weightless inextensible string and swinging to and fro under the action of gravity.

Specific gravity is the ratio of the density of a given material to the density of water.

Speed is rate of motion.

Spin of a gyroscope is its rotation about its axle.

Statics is that part of Mechanics which is concerned with forces which do not produce motion.

Strain is change of size or shape produced by stress.

Strain energy is the capacity of a body for doing work by reason of its being in a state of strain and possessing elasticity.

Strength of centre in simple harmonic motion is the constant ratio of acceleration to displacement.

Stress is a pair of internal forces in the material of a body.

Stress intensity is stress per unit area of cross-section.

Strut is a bar subjected to compressive stress.

Systematic units are those derived in the simplest way from the fundamental units.

Tangential stress is stress produced by parallel forces not in the same straight line.

Tensile stress is the pair of internal forces in the material of a body which is in a state of tension.

Tension is the state of a body which is subject to external pulls, tending to lengthen it in the direction of their action.

Tie is a bar or cord subjected to tensile stress.

Torque (or turning moment) is that which produces, or tends to produce, rotation in the matter upon which it acts.

Torsion is the state of a body which is subject to twisting.

Tractive resistance is the resistance to motion of a vehicle apart from its inertia.

Translation is motion from one place to another.

Units are quantities chosen as standards of comparison.
Vector is a straight line representing a vector quantity.
Vector quantities are those possessing both magnitude and direction.
Vector quantities of the first class are those connected with motion of translation.
Vector quantities of the second class are those connected with motion of rotation.
Velocity is rate of change of position.
Watt is a rate of doing work equal to one joule per second.
Work is equal to the product of force and distance, in the same direction: or of torque and angle turned through.
Young's modulus is the ratio of direct stress to direct strain for a particular material.

DIMENSIONS OF VARIOUS QUANTITIES

Quantity.	Dimensions.		
	Mass.	Length.	Time.
Mass	1	0	0
Length	0	1	0
Time	0	0	1
Area	0	2	0
Volume	0	3	0
Density	1	−3	0
Velocity (Linear) (includes Speed)	0	1	−1
Do. (Angular)	0	0	−1
Acceleration (Linear) (includes Normal Acceleration)	0	1	−2
Do. (Angular)	0	0	−2
Force (includes Total Stress and Weight)	1	1	−2
Torque (Turning Moment)	1	2	−2
Momentum (Linear) (and Impulse)	1	1	−1
Do. (Angular)	1	2	−1
Stress Intensity (also Moduli of Elasticity)	1	−1	−2
Moment of Inertia	1	2	0
Second Moment of Area	0	4	0
Radius of Gyration	0	1	0
Energy (all kinds) and Work	1	2	−2
Power	1	2	−3

The following quantities are purely numerical and their dimensions are therefore, in each case, $[M]^0[L]^0[T]^0$:—

Specific Gravity.
Coefficient of Friction.
Circumference Ratio (π).
Angles.
Velocity Ratio.
Mechanical Advantage.
Efficiency.

LIST OF SYMBOLS (see page 12)

A	Area.
a	Linear Acceleration : also Element of Area : also Length.
b	Breadth.
C	Constant : also Centre of Oscillation.
D	Diameter.
d	Do. : also Depth.
E	Young's Modulus.
e	Strain (tensile or compressive).
F	Force of Friction.
f	Stress Intensity.
G	Modulus of Rigidity : also Centre of Gravity.
g	Acceleration due to Gravity.
h	Height : also Distance.
I	Moment of Inertia : also Second Moment of Area.
i	Do. do. of element.
k	Radius of Gyration.
L	Length of Simple Equivalent Pendulum.
l	Length.
[L]	Dimension of Length.
M	Mass.
m	Do. of element : also Mass per unit length.
[M]	Dimension of Mass.
N	Normal Reaction : also Number of Revolutions per Minute.
O	Centre of Suspension : also Centre of Rotation.
P	Force.
Q	Do.
p	Stress Intensity : also Force acting on Particle.
R	Reaction : also Resultant : also Radius.
r	Radius.
S	Stress (total).
s	Space (distance).
T	Torque : also Tension.
t	Time : also Thickness.
[T]	Dimension of Time.
u	Initial Linear Velocity.
V	Velocity Ratio.
v	Linear Velocity.
W	Weight.
X	Distance of Centre of Gravity or Centroid from Axis.
x	Unknown or Variable Distance : Distance of Element from Axis.
y	Do. do. do. do.
z	Do. do. do. do.
α (alpha)	Angular Acceleration : also Angle (known).
β (beta)	Angle (known).
γ (gamma)	Do. do.

δ	(delta)	Deflection, Distortion, Extension, or Shortening.
η	(eta)	Efficiency.
θ	(theta)	Angle (unknown or variable).
μ	(mu)	Coefficient of Friction: also Strength of Centre.
π	(pi)	Ratio of Circumference of Circle to its Diameter.
ϱ	(rho)	Density.
Σ	(sigma)	Sum of all such quantities as . . .
$\int$	do.	Do. do. do.
ϕ	(phi)	Angle of Friction: also Shear Strain.
ω	(omega)	Angular Velocity.
Ω	do.	Do. do. of Precession.

USEFUL CONSTANTS

$\pi = 3{\cdot}1416 = \frac{22}{7}$ approx.

$\frac{1}{4}\pi = 0{\cdot}7854 = \frac{11}{14}$,,

$\pi^2 = 9{\cdot}8696 = \frac{69}{7}$,,

$\frac{1}{\pi} = 0{\cdot}3183 = \frac{7}{22}$,,

$\frac{1}{\pi^2} = 0{\cdot}1013 = \frac{7}{69}$,,

Values of g.

Locality.	ft./sec.^{2}	cm./sec.^{2}
London	32·19	981
Equator	32·09	978
Poles	32·26	983

Specific Gravities and Densities (approximate)

	Specific Gravity.	Density lbs. per cu. foot.	Density lbs. per cu. inch.
Water	1·0	62·3	·036
Cast-iron	7·2	450	·26
Wrought-iron	7·7	480	·28
Steel (mild)	7·85	490	·285
Aluminium	2·6	162	·095
Brass	8·0	500	·29
Copper	8·8	550	·32
Lead	11·4	712	·42
Oak	·93	58	·034
Ash	·76	47	·027
Yellow Pine	·51	32	·018

Elastic Limits and Moduli (approximate)
(in tons-weight per square inch)

	Young's Modulus.	Mod. of Rigidity.	Elastic Limit. Tension.	Shear.
Cast-iron (aver.)	7,000	2,800	4·5	3·6
Wrought-iron	12,500	5,000	11	9
Steel (mild)	13,000	5,200	15	11
Brass	6,000	2,400	—	—
Copper	7,000	2,800	—	—

METRIC EQUIVALENTS OF BRITISH UNITS

Length.	One foot	= 30·48 cm.	= approx.	$30\frac{1}{2}$ cm.
	One inch	= 2·54 cm.	= ,,	$2\frac{1}{2}$ cm.
	One mile	= 1·609 km.	= ,,	$1\frac{3}{5}$ km.
	One cm.	= ·394 inch	= ,,	$\frac{2}{5}$ inch.
	One metre	= 3·28 feet	= ,,	$3\frac{2}{7}$ feet.
	One kilom.	= ·6214 mile	= ,,	$\frac{5}{8}$ mile.
Mass.	One pound	= 453·6 gm.	= ,,	450 gm.
	One ounce	= 28·35 gm.	= ,,	28 gm.
	One ton	= 1·016 tonne	= ,,	$\frac{60}{59}$ tonne.
	One gramme	= ·0353 oz.	= ,,	$\frac{1}{28}$ ounce.
	One kilog.	= 2·205 lbs.	= ,,	$2\frac{1}{5}$ lbs.
	One tonne	= ·984 ton	= ,,	$\frac{59}{60}$ ton..
Force.	One poundal	= 13,825 dynes	= ,,	14,000 dynes.
	One dyne	= ·0000723 pdl.	= ,,	$\frac{1}{14,000}$ pdl.
Work and Energy.	One ft. pdl.	= ·042 joule	= ,,	$\frac{1}{24}$ joule.
	One ft. lb.	= 13,825 gm. cm.	= ,,	14,000 gm. cm.
	One erg	= ·0000024 ft. pdl.	= ,,	$\frac{1}{420,000}$ ft. pdl.
	One joule	= 2·37 ft. pdls.	= ,,	$2\frac{2}{5}$ ft. pdls.
	One kgm. cm.	= ·0723 ft. lb.	= ,,	$\frac{1}{14}$ ft. lb.
Power.	One H.P.	= 746 watts	= ,,	750 watts.
	One kilow.	= 1·34 H.P.	= ,,	$1\frac{1}{3}$ H.P.

EQUIVALENT VELOCITIES

One mile per hour	=	$\frac{22}{15}$ feet per second
	=	$\frac{33}{38}$ knot
One foot per second	=	$\frac{15}{22}$ mile per hour
One knot	=	$\frac{38}{33}$ miles per hour
One revolution per minute	=	$\frac{11}{105}$ radian per second
One radian per second	=	$\frac{105}{11}$ revolutions per minute

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 4	9 8	13 12	17 16	21 20	26 24	30 28	34 32	38 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 4	8 7	12 11	15 15	19 19	23 22	27 26	31 30	35 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 3	7 7	11 10	14 14	18 17	21 20	25 24	28 27	32 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 3	7 7	10 10	13 12	16 16	20 19	23 22	26 25	30 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 3	6 6	9 9	12 12	15 15	18 17	21 20	24 23	28 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 3	6 5	9 8	11 11	14 14	17 16	20 19	23 22	26 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 3	5 5	8 8	11 10	14 13	16 15	19 18	22 21	24 23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3 2	5 5	8 7	10 10	13 12	15 15	18 17	20 19	23 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 2	5 5	7 7	9 9	12 11	14 14	16 16	19 18	21 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 2	4 4	7 6	9 8	11 11	13 13	16 15	18 17	20 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3		5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

FUNCTIONS OF ANGLES

Angle. Degrees	Angle. Radians.	Chord.	Sine.	Tangent.	Co-tangent.	Cosine.			
0°	0	0	0	0	∝	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	86
5	·0873	·087	·0872	·0875	11·4301	·9962	1·351	1·4835	85
6	·1047	·105	·1045	·1051	9·5144	·9945	1·338	1·4661	84
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	·9903	1·312	1·4312	82
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1763	5·6713	·9848	1·286	1·3963	80
11	·1920	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1·3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2433	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	75
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3·0777	·9511	1·176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	70
21	·3665	·364	·3584	·3839	2·6051	·9336	1·133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
23	·4014	·399	·3907	·4245	2·3559	·9205	1·104	1·1694	67
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1·1519	66
25	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	63
28	·4887	·484	·4695	·5317	1·8807	·8829	1·030	1·0821	62
29	·5061	·501	·4848	·5543	1·8040	·8746	1·015	1·0647	61
30	·5236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·5411	·534	·5150	·6009	1·6643	·8572	·985	1·0297	59
32	·5585	·551	·5299	·6249	1·6003	·8480	·970	1·0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	57
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	56
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	55
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	54
37	·6458	·635	·6018	·7536	1·3270	·7986	·892	·9250	53
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	52
39	·6807	·668	·6293	·8098	1·2349	·7771	·861	·8901	51
40	·6981	·684	·6428	·8391	1·1918	·7660	·845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	48
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	46
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45°
			Cosine	Co-tangent	Tangent	Sine	Chord	Radians (Angle)	Degrees (Angle)

SPECIMEN EXAMINATION PAPERS

These papers are reproduced by permission of Messrs. William Clowes & Sons, Ltd., publishers of the complete sets, and with the sanction of the Institution of Civil Engineers and the Institution of Mechanical Engineers for their respective papers.

The papers are inserted for the information of students, but it may be noted that a few of the questions included in them are outside the scope of this book.

1. INSTITUTION OF CIVIL ENGINEERS, STUDENTSHIP

April, 1922

ELEMENTARY MECHANICS

Not more than EIGHT *questions to be attempted by any Candidate.*

The maximum number of marks obtainable for each question is shown in brackets.

(Gravity acceleration = 32 feet per second per second; 1 cubic foot of water weighs 1,000 ounces.)

1. What is the value of $\frac{1}{2}$ H.P. in ton-inch-hour units. Also calculate the value of 1 H.P. in centimetre-gramme-second units. 1 metre = 39·37 inches. 1 ounce = 28·35 grammes. [20]

2. A square table, with a leg at each corner, has one leg removed; a weight equal to the weight of the table is placed upon it in such a position that the pressure on each of the three remaining legs is equal. Find this position and indicate it by means of a sketch. [25]

3. A man weighing 12 stones carries $1\frac{1}{2}$ cwt. up a 30 feet long ladder, leaning against a wall at an inclination of 30 degrees to the vertical. If he travels from the bottom to the top of the ladder in 1 minute find the horse-power he is exerting during the operation. [20]

4. Three equal poles are arranged to form a tripod, the feet of the poles are apart from each other a distance equal to the length of a pole. If a weight W is suspended from the apex, what is the thrust down each pole? [25]

5. A pendulum 39 inches long oscillates once every second; how many similar oscillations will a pendulum, 42 inches long, make in 1 hour? [20]

6. A circular plate of iron, having a specific gravity of 7·5 and weighing 15 ounces in air, rests at the bottom of a glass of water. What is the pressure on the bottom of the glass due to the iron plate ; also, what would the pressure be if the glass contained oil, having a specific gravity 0·75, instead of water. [20]

7. Twelve pounds of a liquid, having a specific gravity 1·4, are mixed with 7 lbs. of a liquid having a specific gravity of 0·7. What is the specific gravity of the mixture ? [25]

8. Define " Coefficient of Friction " and " Angle of Friction," stating the relationship between them. [20]

9. A body falling freely from rest, passes through 177 feet in the sixth second ; what is the value of gravity in foot-second units ? [20]

10. An uniform gate weighing 500 lbs. is 9 feet long. It is supported by two hinges 3 feet apart, vertically above each other at one end ; the whole of the weight of the gate is carried by the lower hinge. Calculate the forces acting on each of the hinges and sketch their direction so that the condition of equilibrium is proved. [20]

11. Two equal masses, 2 lbs. each, hang over a frictionless and weightless pulley ; a mass of 2 lbs. is laid on one of them. Find the increased pressure on the axle of the pulley. [25]

12. How many cubic feet of water can be raised in 1 hour through a height of 100 feet by a pump developing 90 H.P. ? [20]

2. INSTITUTION OF MECHANICAL ENGINEERS, STUDENTSHIP

April, 1922

ELEMENTARY MECHANICS

(3 *hours allowed.*)

Not more than SIX *questions to be answered.*

The maximum number of marks obtainable for each question is shown in brackets.

1. Assuming the parallelogram of forces, prove that the moment of the resultant of two coplanar, non-parallel, forces about a point in their plane is equal to the algebraic sum of the moments of the two forces about the same point; and deduce the corresponding theorem for parallel forces. [16]

2. Two uniform rods AB, AC, each 12 inches long and 6 lb. in weight, smoothly hinged together at A, and having their ends B, C

connected by an elastic cord whose initial length is 9 inches, and which stretches an extra inch for each 1 lb. tension in it, are supported by a pivot through the mid-point of AB. Find the angle between the rods in the position of equilibrium. [16]

3. Find the distance of the centre of gravity from the parallel sides of a flat metal plate in the form of a trapezium whose parallel sides are respectively 5 feet and 2 feet long, the distance between them being 3 feet. [16]

4. Four equal rods OA, OB, OC, OD, each 6 inches long, of negligible weight, have their lower ends attached to the corners of a square framework ABCD, the bars of which are also 6 inches long. The upper ends are hinged together at O, and have a 10 lb. weight attached there. Find the stresses in the sloping rods, and the consequent tension in the bars of the square frame. [17]

5. A steam crane working at 3 h.p. raises a weight of 11 tons through 45 feet in 20 minutes. Find what percentage of the work is done against friction. [16]

6. An aeroplane which can travel 60 m.p.h. in still air is to travel due north, in a wind which blows from the north-east at 20 miles per hour. Find the direction in which it should be steered, and the speed of its motion over the earth. [17]

7. Two smooth inclined planes of 30° slope have their highest points in contact and their respective lines AB, BC of steepest slope in the same vertical plane. A 10-ton weight on one plane is dragging up a 6-ton weight on the other by means of a rope of negligible weight attached to them and passing over a smooth pulley at the top. Find the acceleration, and the time required to move 100 feet from rest on the planes.

If the planes were rough, find the least coefficient of friction which would prevent motion, assuming the same coefficient for both planes. [17]

8. A 10-ton truck moving at 20 feet per second overtakes a 5-ton truck which was moving in the same direction at 15 feet per second. Assuming that they travel on together after the collision, find their common velocity, and calculate in foot-tons the loss of kinetic energy caused by the collision. [16]

9. A ball is thrown horizontally at 100 feet per second from the top of a vertical cliff 400 feet high. Find how far from the foot of the cliff, and with what velocity, it strikes the ground. [16]

10. A train of 180 tons starting from rest with uniform acceleration, on the level, gets up a speed of 40 m.p.h. in 7 minutes, resistances being 15 lb. per ton. Find the maximum horse-power exerted by the engine. Find also the subsequent horse-power needed to continue at this speed. [17]

3. INSTITUTION OF CIVIL ENGINEERS, ASSOCIATE MEMBERSHIP

April, 1922

MECHANICS

Not more then EIGHT *questions to be attempted by any Candidate.*

The maximum number of marks obtainable for each question is shown in brackets.

1. Discuss the conditions necessary for the equilibrium of a number of coplanar forces, pointing out the difference between the cases of a set of concurrent and non-concurrent forces. [20]

2. A circular plate has a diameter of 3 feet. At a point 1 foot from the centre of the plate is the centre of a hole 5 inches diameter. Find the centre of gravity of the remaining portion of this plate about an axis touching the edge of the large circle and at right angles to a straight line passing through the two centres. [20]

3. The strengths of rectangular beams are proportional to their (breadths) $\times$ (depths)^2 $\div$ spans. Explain this. [20]

4. A lifting block used for heavy weights is hung from the apex of a tripod each of whose legs is 30 feet long, with their lower ends resting at the corners of an equilateral triangle whose sides are 20 feet long. When the lifted load is 10 tons, find the thrust in each leg. [25]

5. Explain with sketches the " method of sections " used for calculating stresses in the members of a framed girder. [20]

6. By " method of sections " find the forces in the bars of the given frame which are cut by the dotted line LM. [25]

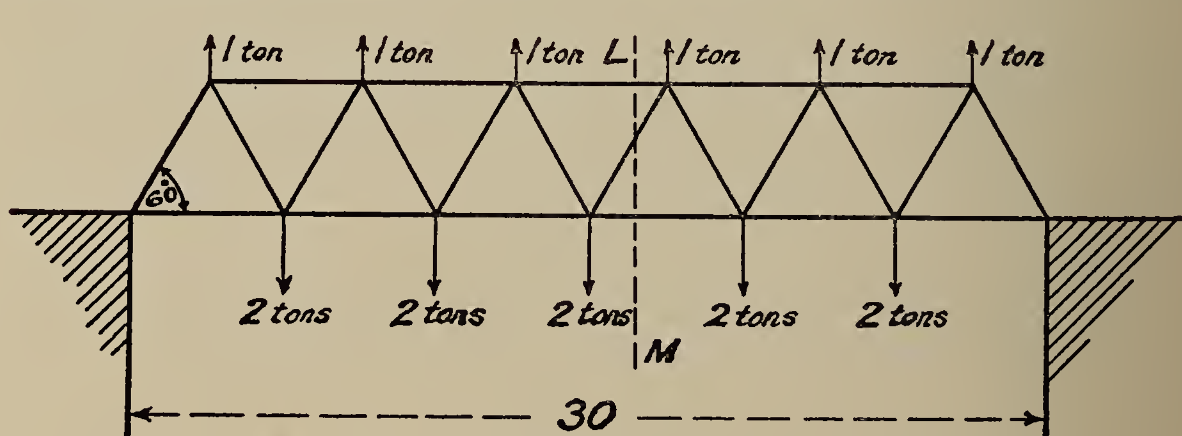


FIG. 165,

7. Find the moment of inertia in an I section whose web is $\frac{1}{2}$ inch thick, the flanges are 8 inches wide and 1 inch thick, and the [20]

total depth outside the flanges is 12 inches. Calculate I about the horizontal and vertical axes XY and AB through the centre of gravity.

8. A caisson is used to close the entrance to a dock—the width at the water surface being 60 feet, and at the bottom 40 feet, while the depth of the water is 30 feet inside the dock. Find the total pressure on the wet face of the caisson and the depth of the centre of pressure below the water surface. [25]

(The weight of the water is to be taken as 64 lbs. per cubic foot.)

9. Explain the meaning of the term "instantaneous centre" and show how it may be used for an ordinary steam engine to find the piston velocity when the speed of the crank pin is known. [20]

10. What is the centrifugal force of a mass rotating about a fixed point? The wheel of a locomotive has a diameter of 4 feet and there is a mass of 5 lbs. attached to a point $1\frac{1}{2}$ foot from the centre. Find the blow upon the rails caused by this mass when the train is running at 60 miles an hour. [20]

11. Define Newton's second law of motion. A man weighing 12 stone stands on the floor of an ascending lift. His pressure on the floor of the lift is 15 lbs. more than his static pressure. Find the upward acceleration of the man which causes the extra pressure. [20]

12. The diameter of the circle in a centrifugal railway is 30 feet; before reaching this the car runs down a slope and is then taken by the rails round the circle. Find the height of the top point of the slope in order that the car may make a complete circle without leaving the rails. [25]

4. INSTITUTION OF MECHANICAL ENGINEERS, ASSOCIATE MEMBERSHIP

April, 1922

APPLIED MATHEMATICS

(3 *hours allowed.*)

Not more than SIX *questions to be answered.*

The maximum number of marks obtainable for each question is shown in brackets.

1. Give the conditions for the equilibrium of a body at rest under any system of coplanar forces.

The jib of a crane makes an angle of 45° with the vertical, and the tension rod from the end of the crane to the fixed upright about

which the jib turns makes an angle of 60° with the vertical. Prove that when lifting a weight W the pressure on the jib is $\frac{W\sqrt{6}}{\sqrt{3}-1}$. [17]

2. State what is meant by the terms coefficient of friction and angle of friction ?

A rough plane is inclined at an angle of 30° to the horizontal, and carries upon it a weight W which is supported by a string parallel to the plane. Prove that the tension in the string is $\frac{W}{2}(1 - \mu\sqrt{3})$ where μ is the coefficient of friction. [16]

3. Draw to scale the bending moment and shearing force diagrams for a beam 30 feet long supported at each end and uniformly loaded with 200 lb. per foot run. [16]

4. A pressure gauge is made from a glass U tube, by filling one side with water (density 1), and the other side with paraffin (density 0·78), the height of water above the common surface is 3·12 inches. Find the height of the paraffin. [16]

5. A circular tank 30 inches diameter and 20 feet high contains crude oil (density 0·85). Find the pressure on the bottom and the intensity of pressure in an outlet pipe in the base. Weight of 1 cubic foot of water 62·5 lb. [16]

6. A circular tank with very thin sides floats with its axis vertical in water, a weight of 200 lb. put into the tank causes it to sink 1·5 inches. Find the diameter of the tank. [16]

7. Define the terms "velocity" and "acceleration," and find the space passed over from rest by a body under a constant acceleration. A stone is dropped into water from a high tower, and the splash is heard 3 seconds after. Find the height if the velocity of sound is 1,100 feet per second and $g = 32$. [17]

8. State and explain what is meant by Newton's laws of motion.

A gun fires a shot of weight W which would penetrate to a depth L into a fixed target. The shot is fired into a target of similar material which is free to move and of a weight three times that of the shot. If the penetration is three-quarters L, show that the shot and target will move with a velocity of one-quarter that of the shot, neglecting heating and friction. [18]

9. A cast-iron flywheel has a rim of mean diameter 11 feet 6 inches, width of rim 20 inches, depth of rim 9 inches. Find the kinetic energy of the wheel at 160 r.p.m. Weight of cast iron is 0·26 lb. per cubic inch. [16]

10. Define the instantaneous centre of a moving body and find its position for the connecting rod of an engine. [16]

BIBLIOGRAPHY

The following list of books is inserted for the benefit of any student who may wish to extend his study of Mechanics in some particular direction, and who desires a suitable text-book. It is not suggested that the list is in any way exhaustive.

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For Laboratory Experiments.

Taylor, "Laboratory Engineering" (University of Lond. Press).

For Calculus.

Thompson, "Calculus Made Easy" (Macmillan & Co.).

For Practical Applications.

Perry, "Applied Mechanics" (Cassell & Co.).

Low, "Applied Mechanics" (Longmans, Green & Co.).

Goodman, "Mechanics Applied to Engineering" (Longmans, Green & Co.).

ANSWERS TO EXAMPLES

Examples I (pages 22–4)

2. 54·3 ft. per sec.
3. 49·8 lbs. per cu. ft.
4. The car: $\frac{2}{3}$ ft. per sec.
5. 7·21.
6. 12,600 seconds.
7. 1,400 lbs. per sec.: 1,346 cu. ft.
8. 929.
9. 28,315.
10. 25,740 feet.
11. 51·6 miles per hour: 75·7 feet per sec.
12. 496·5 lbs. per cu. ft.
13. $[L][T]^{-1}$.
15. ·311 lb. per sec.
16. $\frac{1}{2}$ inch.
17. 2,835 pounds.
18. 10·95 seconds.
19. $224\frac{1}{2}$ pounds per sq. ft.
20. (a) $[M][L]^{-1}$.
(b) $[L][T]^{-1}$.
(c) $[M][L]^{-3}$.
(d) $[L]^{3}$.
(e) $[M][L][T]^{-1}$.
21. $173\frac{3}{4}$ miles.
22. (a) $19\frac{1}{4}$ miles per hour.
(b) 28·2 feet per sec.
23. 2,830 cm. per sec.
24. $[L]^{-1}$.
25. 13·1 miles per hour.

Examples II (pages 37–42)

5. 51·2 poundals at an angle of 35° with smaller force.
6. 5·48 tons-weight at an angle of $29\frac{1}{2}$° with larger force.
7. 41·4 pounds-weight at an angle of 50° with smaller force.
8. A downward force of 22·8 poundals between P and Q at an angle of 50° from P.
9. 451 poundals at an angle of 37° with larger force.
10. 15·8 pounds-weight horizontal: 9·0 pounds-weight vertical.
11. 7·96 tons-weight: 7·26 tons-weight.

12. Active force of 140 lbs.-wt. vertically downwards on table : passive force of 140 lbs.-wt. vertically upwards on man.
13. 42·8 pounds-weight at an angle of 12° with larger force.
14. (**a**) 7·5 poundals. (**b**) 73·4 poundals.
15. 350 poundals at an angle of 52° from horizontal.
16. 99·4 pounds-weight.
17. 946 lbs.-wt. vertical : 347 lbs.-wt. horizontal.
18. 237 poundals : 255 poundals.
19. 48·2 lbs.-wt. vertical : 55·0 lbs.-wt. horizontal.
20. Q = 96 poundals at an angle of $156\frac{1}{2}°$ with vertical : resultant = 174 poundals at an angle of 139° with vertical.
21. 6 lbs.-wt. downwards at an angle of 11° with vertical.
22. (**a**) 19·1 lbs.-wt. : (**b**) 3·0 lbs.-wt.
23. (**a**) 63 poundals, left to right : (**b**) 41 poundals, downwards.

Examples III (pages 57–62)

6. 798 lb.-ft.
7. (**a**) 156 poundal-ft. (**b**) 784 lb.-ft. (**c**) 343 lb.-ft.
8. 280 pound-feet.
9. 3,200 pound-feet-square.
10. Second force : 615 lb.-ft.
11. 14·86 lbs.-wt.
12. 930 feet^4.
13. 153·1 ton-feet-square.
14. (**i**) [M][L] : (**ii**) $[L]^3$: (**iii**) $[M][L]^2$: (**iv**) $[L]^4$.
15. 5·6 ft. : 294 lb.-ft.
16. $1\frac{1}{2}$ tons-weight.
17. 4·41 tons-wt. at A : 6·09 tons-wt. at B.
18. 38·2 tons-wt. at A : 32·6 tons-wt. at B.
19. 974 lbs.-wt. : 566 lbs.-wt.
20. 33,500 pdl.-ft. : 27,720 pdls.
21. − 2,767·5 pdl.-ft. : − 2,227·5 pdl.-ft. : + 234 pdl.-ft. : − 306 pdl.-ft. : − 2,533·5 pdl.-ft.
22. $3\frac{3}{4}$ ton-feet.
23. $957\frac{1}{4}$ pound-feet-square.
24. 109·2 poundal-feet.
25. 170 pounds-weight.
26. 6·97 feet : 329 pounds-weight.
27. 37·85 pound-feet-square.

Examples IV (pages 73–80)

7. 23° : $59\frac{1}{2}°$.
8. 37 lbs.-wt. : 42 lbs.-wt.
9. 79 poundals : $122\frac{1}{2}$ poundals.
10. 4·93 lbs.-wt. at $30\frac{1}{2}°$ from horizontal.
11. 2·26 lbs.-wt. : 6·79 lbs.-wt. upwards at $103\frac{1}{2}°$ from AB.

12. $87\frac{1}{2}°$: $138°$: $134\frac{1}{2}°$.
13. 55·1 lbs.-wt.: 449 lbs.-wt.
14. 5·82 tons-wt.: 5·52 tons-wt. at $111\frac{1}{2}°$ from AB.
15. 21 lbs.-wt.: 38·1 lbs.-wt. at $66\frac{3}{4}°$ from BC.
16. 130 lbs.-wt.: 93 lbs.-wt. at 1° downwards from horizontal.
17. 24·2 lbs.-wt.
19. 224 lbs.-wt. due west.
20. 108 lbs.-wt.: 225 lbs.-wt. at 20° from vertical.
21. $27\frac{1}{4}$ cwt.: $18\frac{1}{4}$ cwt. upwards at 42° from horizontal.

Examples V (**pages 94–8**)

5. In the brass length, $2\frac{1}{2}$ inches from the lead.
6. 10′ 10″ from centre of smaller ball.
7. 2·16 feet from A.
8. 2·65 inches from C.
9. 1·85 inches: 1·95 inches.
10. 1·98 inches from short edge.
11. ·146 inch from centre of circle, on opposite side from triangle.
12. On diagonal, 3·28 inches from outside corner of smaller cube.
13. 2·05 inches from A: 1·95 inches from B.
14. On axis of cylinder, 1·92 inches from cube.
15. 2·95 inches: 2·90 inches.
16. 2 feet: 1 ft. 11 ins.
17. 9·11 feet: 7·61 feet.
18. On diagonal of cube, at 4·83 inches from corner opposite removed cube.
19. 3·25 inches from A: 2·45 inches from B.
20. 3·35 inches above centre of base.
21. In copper: 4·12 inches: 4·85 inches.
22. 1·53 inches from AB: 2·89 inches from AF.

Examples VI (**pages 113–9**)

7. 66·6 feet per second.
8. 8·5 knots: 42° 9′ west of south.
9. 18·4 miles per hour: 34·4 miles per hour.
10. 7·83 feet per second: 52 feet per second: 3,120 feet.
11. ·321 foot per sec. per sec.
12. 38 miles per hour: 16° 12′ south of west.
13. 68·6 seconds: 113·4 feet: 2·46 feet per second.
14. 2·70 miles.
15. 132·8 feet per second: 4·13 seconds.
16. 34·4 seconds: 738 feet.
17. 50·3 miles per hour, at angle of 2° 17′ with direction in which passenger is walking.
18. 4·315 seconds: 224·3 feet.
19. 582 feet.

20. 7·9 miles per hour : 34° 41′ north of east.
21. Upstream, at angle of 47° with river : $67\frac{1}{4}$ seconds.
22. 6·73 miles per hour in direction at 31° 20′ from north-south line,
23. 121 feet : the dropped stone, by 1·24 seconds.
24. 83·8 feet per second.
25. 35·0 miles per hour : 17·7 m.p.h. : 16·8 seconds.
26. 713 feet : 1,790 feet from A.
27. 194·3 feet : 4·57 seconds.

Examples VII (pages 133–8)

4. 29·3 radians per second : 217 cm. per sec.
5. 24 radians per sec. : 229 revs. per minute.
6. 4·59 feet.
7. 340 revs. per min.
8. $\frac{1}{50}$ radian per second.
9. 31·6 radians per second.
10. 2·47 miles per hour.
11. 1·48 radians per second per second.
12. ·305 radian per second per second.
13. 24·64 radians per second : 19·36 radians per second : ·211 radian per second per second.
14. 44·8 cm. per sec. : 21·36 cm. per sec. per sec. at angle of 17° 42′ with radius.
15. ·147 ft. per sec. per sec. : ·000122 radian per sec. per sec. : ·403 ft. per sec. per sec.
16. 10,850 feet per sec. per sec.
17. 678·5 ft. per sec. per sec.
18. 2·93 feet : 80·3 feet per sec. : 27·4 radians per sec.
19. 42·1 ft. per sec. per sec. at an angle of $17\frac{1}{2}$° with larger acceleration.
20. 1,153 revs. per min. : 38° 40′.
21. 3·9 feet.
22. 12·8 revs. per min. : 2·21 miles per hour.
23. 7·38 radians per sec. per sec.
24. 29·4 radians per sec. : 0° 10′ from horizontal.

Examples VIII (pages 153–9)

5. 28,350 F.P.S. units.
6. 75·6 million C.G.S. units.
7. 13·55 million F.P.S. units, due north.
8. 30·7 seconds.
9. 560 poundals.
10. 54·7 feet per sec. : 1·8 million F.P.S. units.
11. 2·18 tons-weight.
12. 25·3 pounds-weight : 8,010 feet.
13. 4·88 tons-weight.

14. 5·61 pounds: 162·8 poundals: 149 poundals: $152\frac{1}{2}$ poundals.
15. ·0595 foot per sec. in direction of second force: 300 F.P.S. units.
16. ·47 ft. per sec. per sec.: 183·5 seconds.
17. (i) 4·06. (ii) 8·62.
18. (a) 21·9 F.P.S. units in direction $30\frac{1}{2}°$ from horizontal: (b) 29·1 units in direction $49\frac{1}{2}°$ from horizontal.
19. 13,840.
20. 66,000 F.P.S. units in direction $13\frac{1}{2}°$ south of east.
21. 1·56 feet per sec.
22. 29,333 poundals.
23. 4·1 tons-weight.
24. ·0313 foot per sec. per sec.: 42 minutes: 78·8 ft. per sec.

Examples IX (pages 169–177)

8. ·255.
9. 123 poundals.
10. 51·2 pounds.
11. (a) 143 poundals. (b) 426 poundals.
12. 898,000 dynes.
13. ·217.
14. 13·9 poundals.
15. ·246.
16. 145 pounds-weight.
17. 4·335 tons-weight.
18. 11·93 feet.
19. 97·7 miles per hour.
20. 31° 50′.
21. 10·44 feet: 9·75 feet.
22. 5 mins. 20 secs.: 2·00 miles: 19,800 foot-ton-second units.
23. 2·645 pounds-weight.
24. 493 feet: 84 seconds.
25. 12·12 lbs.-wt.: 43·7 lbs.-wt. at angle of 16° 5′ with vertical: ·288.

Examples X (pages 192–9)

7. 6·15 tons-weight per square inch.
8. 15,700 pounds-weight per square inch.
9. 10·73 tons-weight.
10. 6·05 tons-weight.
11. 1·037 inches.
12. 1·59 inches.
13. ·1375 inch.
14. ·0996 inch.
15. ·000474.
16. 6·21 tons-weight per square inch: ·000497.
17. 7·23 tons-weight per square inch: ·000566: ·0878 inch.
18. ·174 inch: 14,740 pounds-weight.

19. 3·82 tons-weight per square inch : ·000294 : 13,030 tons-weight per square inch.
20. 73·5 tons-weight.
21. 1·47 inches.
22. ·000519.
23. 4,990 tons-weight per square inch.
24. 7,000 tons-weight per square inch.
25. $\frac{3}{8}$ inch.
26. Stresses : AD, + 15·0 cwt. : DC, + 11·0 cwt. : CB, + 12·1 cwt. : DE, + 6·93 cwt. : AE, — 13·0 cwt. : EB — 6·06 cwt. : EC, — 6·93 cwt.
27. Reactions : 6·31 cwt. each. Stresses : AB, + 6·86 cwt. : BC, + 3·65 cwt. : CD, + 6·86 cwt. : EA, — 3·43 cwt. : EB, — 0·44 cwt. : EC, — 0·44 cwt. : ED, — 3·43 cwt.

Examples XI (pages 215–223)

7. 1,076 poundals.
8. 1·596 feet per sec. per sec. : 4·165 tons-weight.
9. 510 pounds.
10. 3,022 cm. per sec.
11. 116,200 poundals.
12. 1·375 inches.
13. **(i)** 3·9 poundals : **(ii)** 32·1 poundals : 44·25 revs. per min.
14. 7·78 inches.
15. 20° : 0·33 inch decrease : 0·38 inch increase.
16. 4·1 inches.
17. 50·5 km. per hour.
18. 32°.
19. 51·2 miles per hour.
20. 19$\frac{3}{4}$ inches.
21. 20·6 revs. per minute.
22. 208 revs. per minute.
23. 13 pounds-weight : 15 pounds-weight.
24. 112·8 poundals.
25. 159 revs. per minute : 323 poundals.
26. 83,800 poundal-feet each : 22·5 miles per hour.
27. 34·2 feet.

Examples XII (pages 243–9)

6. 165 foot-poundals.
7. 3,960 foot-pounds.
8. 76·5 foot-tons.
9. 446,000 foot-tons : 943 H.P.
10. 23,600 joules.
11. 6,820 foot-poundals.
12. **(a)** 1,420 foot-poundals : **(b)** 1,055 foot-poundals.

13. 9·66 foot-tons.
14. 202,500 foot-poundals.
15. (**a**) Nil : 987,000 foot-poundals. (**b**) 379,000 foot-poundals : 608,000 foot-poundals. (**c**) 973,500 foot-poundals : 13,500 foot-poundals.
16. 802·5 H.P. : ·073.
17. (**a**) 6·4 H.P. : (**b**) 2·5 H.P.
18. 31·4 foot-tons : 56·6 foot-tons.
19. 1·24 foot-pounds.
20. 798 pounds-weight : ·00058 second.
21. 484 foot-pounds : ·27 pound per cubic inch.
22. 705 pounds-weight : 696 feet per second.
23. 5·67 tons-weight.
24. 2·39 miles.
25. (**a**) 31·5 : (**b**) 19·4 : (**c**) 61·6 per cent.
26. (**a**) 16·85 : (**b**) 34·9 : (**c**) 1,200 pounds-weight.

Examples XIII (pages 269–274)

5. 21,200 pound-feet-square : 399,000 F.P.S. units : 3,860,000 foot-poundals.
6. 116·5 pound-feet-square : 1·25 feet : 246 revs. per minute.
7. 16·25 radians per sec. : 30·0 lb.-ft.-sq. : 46·9 pounds.
8. 9·91 pound-feet : 8·33 foot-tons.
9. 2·64 feet : 410 revs. per minute.
10. 217,500 F.P.S. units : 42·3 foot-tons.
11. 54·2 pounds-weight.
12. 66·6 foot-tons.
13. 66·6 pounds-weight.
14. 1,724 F.P.S. units : 431 F.P.S. units.
15. 24·64 pounds-weight : 6·16 pound-feet.
16. 18·7 kgm. : 424 million C.G.S. units.
17. 91·2 kgm.-cm. : 17·9 radians per second.
18. 142·7 foot-tons : 968 revs. per minute.
19. 3·614 foot-tons.
20. 3,872 pound-feet-square.
21. 688 foot-poundals : 23·8.

Examples XIV (pages 300–3)

3. 764 pound-feet-square.
4. $\frac{2}{3}Ma^2$.
5. $\frac{1}{6}Ma^2$.
6. $\frac{1}{4}bh^3$.
7. $\sqrt{3}.a^4/96$.
8. $\sqrt{3}.a^4/64$.
9. ·853 pound-foot-square : ·341 pound-foot-square.
10. $Ma^2/24$.

11. $\frac{3}{10}Mr^2$.
12. 1,125 inches^4.
13. 80 inches^4.
14. $\pi\varrho r^4(\frac{1}{2}l + \frac{8}{15}r)$.
15. $\pi\varrho r^2(\frac{8}{15}r^3 + \frac{1}{2}r^2 l + \frac{1}{6}r^2 l + \frac{1}{12}l^3)$.
16. 40,850 gm.-cm.-sq.
17. 40·7 feet^4: 26·0 feet^4: 18·1 feet^4.
18. $\frac{2}{3}Ma^2$.
19. $\frac{1}{12}b\varrho(a^4 - 3\pi r^4)$.
20. 12·20 pound-inches-square.
21. 48·81 pound-inches-square.
22. 2·58 inches: 1·63 inches.
23. 9·90 inches: 8·08 inches.
24. 2·02 feet: 1·64 feet: 1·35 feet.
25. $1{\cdot}253a^5\varrho$.
26. 11,870 pound-feet-square.
27. 25,390 pound-feet-square.
28. 115 pound-inches-square: ·23 per cent.

Examples XV (pages 321–5)

6. 2·785 feet: 3·22 units: 8·97 feet per sec. per sec.
7. 1·094 seconds: 3·83 feet per second.
8. 5·0 feet: 3·63 seconds.
9. 5·57 inches.
10. ·68 second: 1·73 feet per second.
11. ·54 second.
12. 67·5 vibrations per minute.
13. 153 cm.
14. 32·0 feet per sec. per sec.
15. 1·336 seconds: 17·48 inches.
16. 1·049 seconds.
17. ·316 per cent.
18. 6·95 inches from centre of rod.
19. 93·75 oscillations per minute.
20. 96·6 foot-poundals: 53·4 foot-poundals.
21. 125 foot-poundals.
22. 18·7 feet: 14·75 feet per sec. per sec.

Examples XVI (pages 342–5)

3. 40·9 pound-feet.
4. ·346 rev. per minute.
5. 9·98 revs. per minute.
6. 392 pound-feet.
7. 1·593 pound-feet-square.
9. 11·44 pound-feet.

12. 166·4 pound-feet.
13. 241 poundal-feet.
14. ·00348 radian per second.
15. 38·3 pound-inches-square.

Examples XVII (pages 353–8)

5. 52·6 revs. per minute: 91·1 revs. per minute.
6. 237 foot-pounds.
7. 4·37 cm.
8. 51·0 miles per hour: ·0934 ft. per sec. per sec.
9. 3,970,000 gramme-centimetres-square.
12. 1,181 kilogrammes-weight per sq. cm.
14. $\sqrt{\frac{2}{5}\left(\frac{R^5-r^5}{R^3-r^3}\right)}$
15. 504 cm. per sec. per sec. at an angle of 4° 33′ with radius.
16. To A, and to a point 3′ 6″ from B.
17. ·0602 inch.
18. 1,782 feet per second: 98,010 foot-poundals.
19. 259·4 cm.
20. 51·8 pounds-weight: 95·0 foot-tons.
21. 5 feet $10\frac{5}{8}$ inches.
22. 7,424,000 foot-poundals: 486,500 F.P.S. units.
23. 22·0 feet per sec.: 329·5 poundals: 56·6 poundals.

INDEX

(References are to **pages** : principal references are given in **heavy type** : those to examples in *italics*).